

# “THE OPTIVERSE” AND OTHER SPHERE EVERSIONS

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ABSTRACT. For decades, the sphere eversion has been a classic subject for mathematical visualization. The 1998 video *The Optiverse* shows geometrically optimal eversions created by minimizing elastic bending energy. We contrast these minimax eversions with earlier ones, including those by Morin, Phillips, Max, and Thurston. The minimax eversions were automatically generated by flowing downhill in energy using Brakke’s Evolver.

## 1. A HISTORY OF SPHERE EVERSIONS

To evert a sphere is to turn it inside-out by means of a continuous deformation, which allows the surface to pass through itself, but forbids puncturing, ripping, creasing, or pinching the surface. An abstract theorem proved by Smale in the late 1950s implied that sphere eversions were possible [Sma], but it remained a challenge for many years to exhibit an explicit eversion. Because the self-intersecting surfaces are complicated and nonintuitive, communicating an eversion is yet another challenge, this time in mathematical visualization. More detailed histories of the problem can be found in [Lev] and in Chapter 6 of [Fra].

The earliest sphere eversions were designed by hand, and made use of the idea of a halfway-model. This is an immersed spherical surface which is halfway inside-out, in the sense that it has a symmetry interchanging the two sides of the surface. If we can find a way to simplify the halfway-model to a round sphere, we get an eversion by performing this simplification first backwards, then forwards again after applying the symmetry. The eversions of Arnold Shapiro (see [FM]), Tony Phillips [Phi], and Bernard Morin [MP] can all be understood in this way.

In practice, two kinds of halfway-models have been used, shown in Fig. 1. The first, used by Shapiro and by Phillips (see Fig. 2, left), is a Boy’s surface, which is an immersed projective plane. In other words it is a way of immersing a sphere in space such that antipodal points always map to the same place. Thus there are two opposite sheets of surface just on top of each other. If we can succeed in pulling these sheets apart and simplifying the surface to a round sphere right-side-out, then pulling them apart the other way will lead to the inside-out sphere.

The other kind of halfway-model is a Morin surface; it has four lobes, two showing the inside and two the outside. A ninety-degree rotation interchanges the sides, so the two halves of the eversion differ by this four-fold

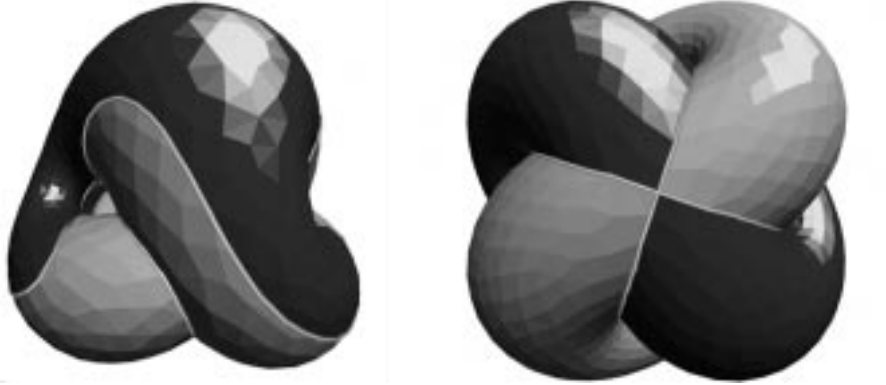


FIGURE 1. These are the halfway-models for the first two minimax eversions. The Boy's surface (left), an immersed projective plane with three-fold symmetry and a single triple point, minimizes Willmore's elastic bending energy. The figure actually shows an immersed sphere, double covering Boy's surface, with the two (oppositely-oriented) sheets pulled apart slightly. The Morin surface shown (right) also minimizes Willmore energy; it has a four-fold rotational symmetry which reverses orientation, exchanging the lighter and darker sides of the surface.

twist. Morin (whose blindness incidentally shows that mathematical visualization goes well beyond any physical senses) and Apéry [Apé] have shown that an eversion based on a Morin surface halfway-model has the minimum possible number of topological events.

Models of this eversion were made by Charles Pugh, and Nelson Max digitized these models and interpolated between them for his famous 1977 computer graphics movie "Turning a Sphere Inside Out" [Max], a frame of which is shown in Fig. 2. Morin eversions have also been implemented on

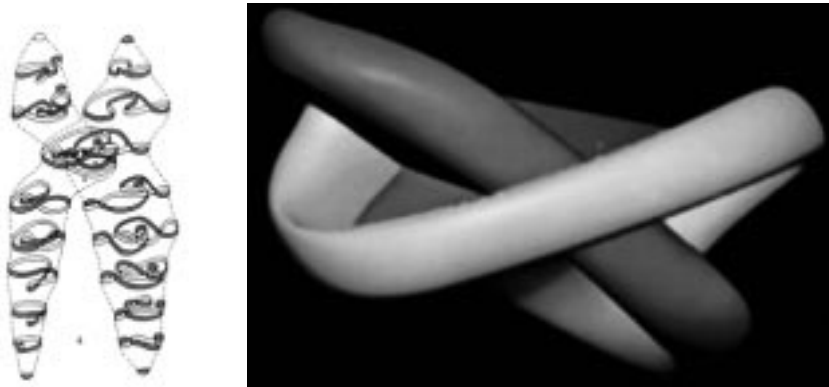


FIGURE 2. This drawing (left) by Tony Phillips [Phi] shows one stage of his sphere eversion based on a Boy's surface halfway-model. This frame (right) from Nelson Max's classic computer animation of a Morin eversion shows a stage near the halfway-model.



FIGURE 3. Still pictures from the Morin eversion implementations by Robert Grzeszczuk (left) and John Hughes (right).

computers by Robert Grzeszczuk and by John Hughes (see Fig. 3), among others. These eversions have usually been laboriously created by hand, splining or interpolating between key frames, or looking for algebraic or trigonometric equations.

A new proof of Smale’s original theorem, providing more geometric insight, was developed by Bill Thurston (see [Lev]). His idea was to take any continuous motion (homotopy) between two surfaces, and make it regular (taking out any pinching or creasing) when this is possible by adding corrugations in a controlled way; the corrugations make the surface more flexible. This kind of sphere eversion was beautifully illustrated in the computer-graphics video “Outside In” [LMM] produced at the Geometry Center in 1994 (see Fig. 4). The corrugation idea is quite natural, and provides a way to understand all regular homotopies, not just the sphere eversion. However, the resulting eversion, though nicely full of symmetry (except, interestingly, the temporal symmetry seen in all sphere eversions based on

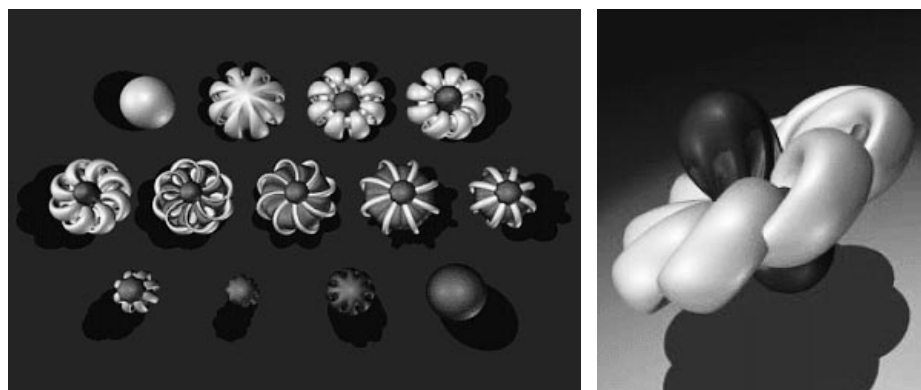


FIGURE 4. These pictures, from “Outside In”, show Thurston’s sphere eversion, implemented through corrugations.

halfway-models), is quite elaborate, and has many more topological events than necessary.

## 2. BENDING ENERGY AND THE MINIMAX EVERSIONS

Our minimax sphere eversions differ from the earlier ones mentioned above in that they are computed automatically by a process of energy minimization. The elastic bending energy for a stiff wire is the integral of squared curvature. For a surface in space, at each point there are two principal curvatures, and their average, the mean curvature, shows how much the surface deviates from being minimal. The integral of squared mean curvature is thus a bending energy for surfaces, often called the Willmore energy [Wil]. (Mathematically this is equivalent to many other formulations because of the Gauss-Bonnet theorem.)

Among all closed surfaces, the round sphere minimizes this bending energy. (The energy is scale-invariant, and we normalize so the sphere has energy 1.) It is also known [LY, Kus] that any self-intersecting surface with a  $k$ -tuple point has energy at least  $k$ . To evert a sphere, it is necessary to pass through some stage with a quadruple point [BM, Hug], and hence energy at least 4.

Francis and Morin realized that the Morin surface and Boy's surface are just the first in an infinite sequence of possible halfway-models for what they call the tobacco-pouch eversion (see Fig. 5, taken from [Fra]). In the 1980's, Robert Bryant [Bry] classified all critical points for the Willmore energy among spheres; they all have integer energy values which (except for the round sphere) are at least 4. Rob Kusner, being familiar with both of these results, realized that he could find, among Bryant's critical spheres, ones with the tobacco-pouch symmetries. Among surfaces with those symmetries, Kusner's have the least possible Willmore energy.

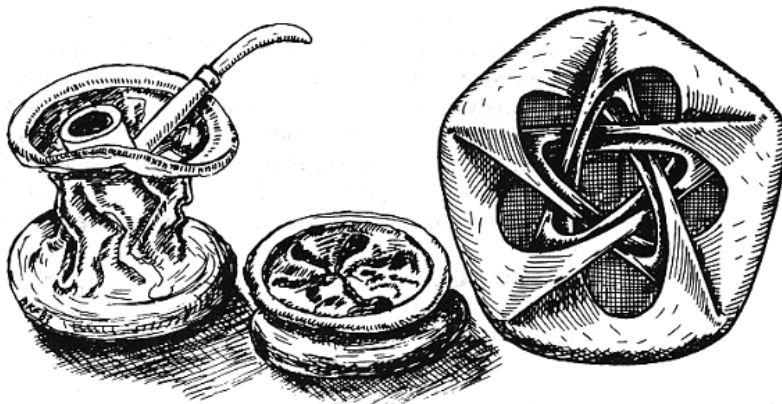


FIGURE 5. These sketches by George Francis show a French tobacco pouch, and a cutaway view of the halfway-model for the order-five tobacco-pouch eversion.

In particular, Kusner’s Morin surface with four-fold orientation-reversing symmetry has energy exactly 4. If we don’t enforce this symmetry, then presumably this surface is an unstable critical point—a saddle point for the energy. Pushing off from this saddle in one downhill direction, and flowing down by gradient descent, we should arrive at the round sphere, since it is the only critical point with lower energy. (Of course, there’s not enough theory for fourth-order partial differential equations for us to know in advance that the surface will remain smooth and not pinch off somewhere.)

As we saw before, such a homotopy, when repeated with a twist, will give us a sphere eversion. This eversion starts at the round sphere (which minimizes energy) and goes up over the lowest energy saddle point; then it comes back down on the other side arriving at the inside-out round sphere.

Around 1995, in collaboration with Kusner and Francis, I computed this minimax eversion [FSK<sup>+</sup>], using Ken Brakke’s Evolver [Bra], which is a software package designed for solving variational problems, like finding the shape of soap films or (see [HKS]) minimizing Willmore energy. We were pleased that the computed eversion was not only geometrically optimal in the sense of requiring the least bending, but also topologically optimal, in that it was one of the Morin eversions with the fewest topological events.

Computations of the higher-order minimax sphere eversions [FSH] (like the one with a Boy’s surface halfway-model) had to wait until Brakke and I implemented some new symmetry features in the Evolver (see [BS]). A minimax tobacco-pouch eversion, whose halfway-model has  $2k$ -fold symmetry, will break some of this symmetry. But it will maintain  $k$ -fold symmetry throughout, and the Evolver now works with only a single fundamental domain for this symmetry. We can find the initial halfway-models by minimizing bending energy while enforcing the full symmetries.

Alternatively, we can compute them directly. Bryant’s classification says that all critical spheres are obtained as conformal transformations of minimal surfaces, and Kusner gave explicit Weierstrass parameterizations for the minimal surfaces he needed. In Fig. 6 we see the minimal surface with four flat ends which transforms into our Morin halfway-model.

The Evolver works with triangulated approximations (with a few thousand triangles) to the true smooth surfaces, and we update the triangulation as needed to maintain a good approximation. Initially, it’s necessary to use the second-order Hessian methods implemented in the Evolver to push off the saddle point and find our way downhill.

In 1998, working with Francis and Stuart Levy, I produced a computer graphics video, “The Optiverse” [SFL], which shows the first four minimax eversions.<sup>1</sup> It premiered at the International Congress of Mathematicians in Berlin, and was also shown at SIGGRAPH and written up nicely in Science News [Pet].

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<sup>1</sup>This video was produced at the NCSA at the University of Illinois. More information about it is available on our website at <http://new.math.uiuc.edu/optiverse/>.

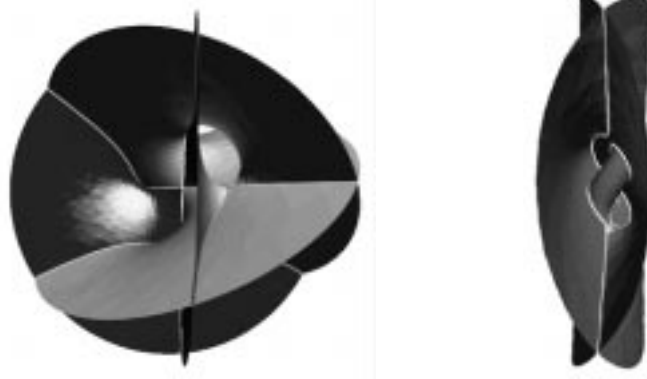


FIGURE 6. This minimal surface (left), with four flat ends, gives rise to Kusner's Morin surface of least Willmore bending energy, when a conformal Möbius transformation is applied to compactify it. If the transformation sends the double-tangent point to infinity, we get an interesting picture (right).

Many scenes in the video (as in Fig. 7) capture views of the eversion also available in our real-time interactive computer animation running on desktop workstations or immersive virtual-reality environments. These show a solid, colored surface, with white tubes around the double-curves of self-intersection. Here we are aware of the triangulation used for the computation, especially if we leave gaps between triangles or in their middles.

But no one method of rendering can give all useful visual information for a nonintuitive phenomenon like a self-intersecting surface. So other parts of the video (like Fig. 8) were rendered as transparent soap films, with the soap bubble shader I wrote for Renderman (see [AS]).

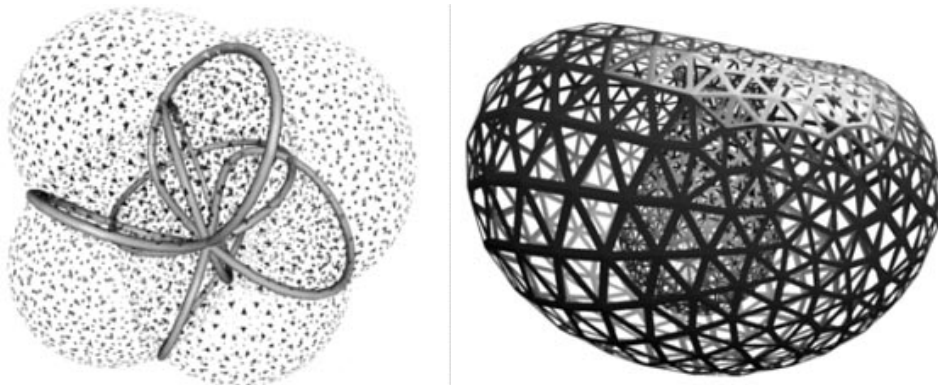


FIGURE 7. The Morin halfway model (left), shown with all triangles shrunk, has an elaborate set of double curves where the surface crosses itself. An late stage in the eversion is like a gastrula (right), shown here as a triangular framework.

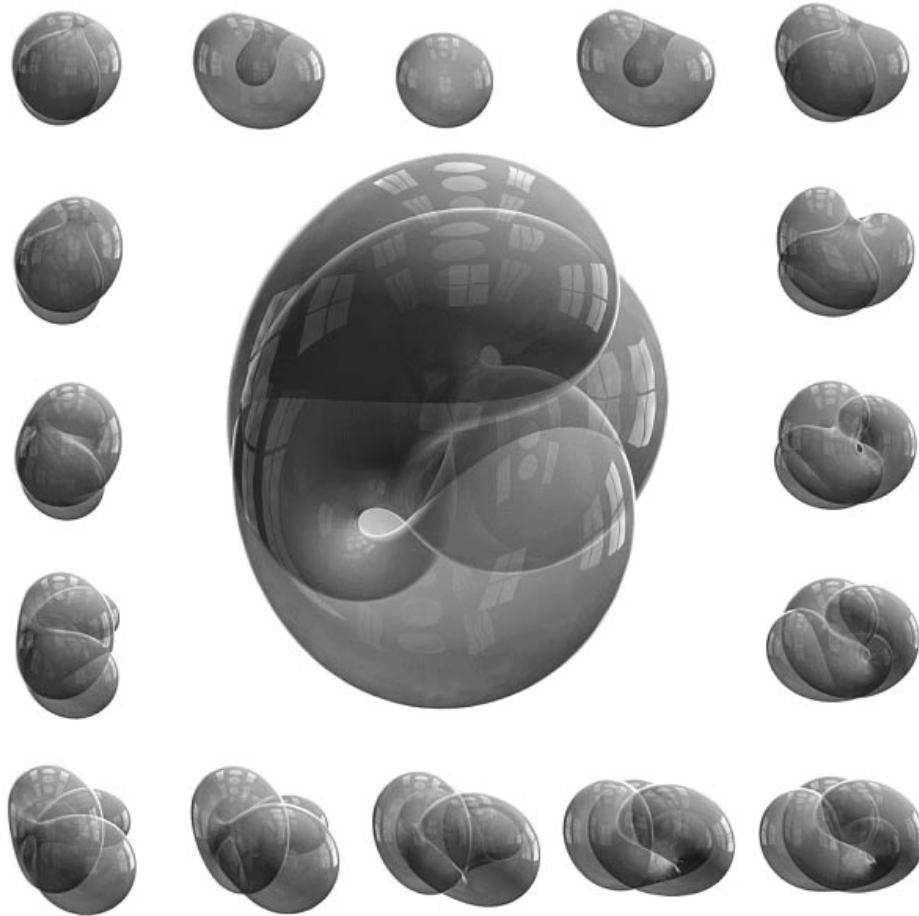


FIGURE 8. This minimax sphere eversion is a geometrically optimal way to turn a sphere inside out, minimizing the elastic bending energy needed in the middle of the eversion. Starting from the round sphere (top, moving clockwise), we push the north pole down, then push it through the south pole (upper right) to create the first double-curve of self-intersection. Two sides of the neck then bulge up, and these bulges push through each other (right) to give the second double-curve. The two double-curves approach each other, and when they cross (lower right) pairs of triple points are created. In the halfway-model (bottom) all four triple points merge at the quadruple point, and five isthmus events happen simultaneously. This halfway-model is a symmetric critical point for the Willmore bending energy for surfaces. Its four-fold rotation symmetry interchanges its inside and outside surfaces. Therefore, the second half of the eversion (left) can proceed through exactly the same stages in reverse order, after making the ninety-degree twist. The large central image belongs between the two lowest ones on the right, slightly before the birth of the triple points.

### 3. TOPOLOGICAL STAGES IN THE EVERSION

A generic regular homotopy has topological events occurring only at isolated times, when the combinatorics of the self-intersection changes. These events happen at times when the surface normals at a point of intersection are not linearly independent. That means either when there are two sheets of the surface tangent to each other (and a double curve is created, annihilated, or reconnected), or when three intersecting sheets share a tangent line (creating or destroying a pair of triple points), or when four sheets come together at a quadruple point.

The double tangency events come in three flavors. These can be modeled with a rising water level across a fixed landscape. As the water rises, we can observe creation of a new lake, the conversion between a isthmus of land and a channel of water, or the submersion of an island. These correspond to the creation, reconnection, or annihilation of a double-curve, here seen as the shoreline.

In the minimax eversion with two-fold symmetry, seen in Fig. 8 and Fig. 9, the first two events create the two double-curves. When these twist around to intersect each other, two pairs of triple points are created. At the halfway stage, six events happen all at the same time. Along the symmetry axis, at one end we have a quadruple point, while at the other end we have a double tangency creating an isthmus event. Finally, at the four “ears”, at the inside edge of the large lobes, we have additional isthmus events: two ears open as the other two close. (See [FSK<sup>+</sup>] for more details on these events.)

The three-fold minimax eversion, using the Boy’s surface of Fig. 1 as a halfway-model, has too many topological events to describe easily one-by-one, and its three-fold symmetry means that the events are no longer all generic. But we can still follow the basic outline of the eversion from Fig. 10.

#### ACKNOWLEDGMENTS

The minimax sphere eversions described here are work done jointly in collaboration with Rob Kusner, Ken Brakke, George Francis, and Stuart Levy, to whom I owe a great debt. I would also like to thank Francis, Robert Grzeszczuk, John Hughes, AK Peters, Nelson Max and Tony Phillips for permission to reproduce figures from earlier sphere eversions. This paper will appear in the proceedings of two summer 1999 conferences on Mathematics and Art: ISAMA 99 (June, San Sebastián, Spain), and Bridges (July, Winfield, Kansas). I wish to thank Nat Friedman, Reza Sarhangi, and Carlo Séquin for the invitations to speak at these conferences. My research is supported in part by NSF grant DMS-97-27859.



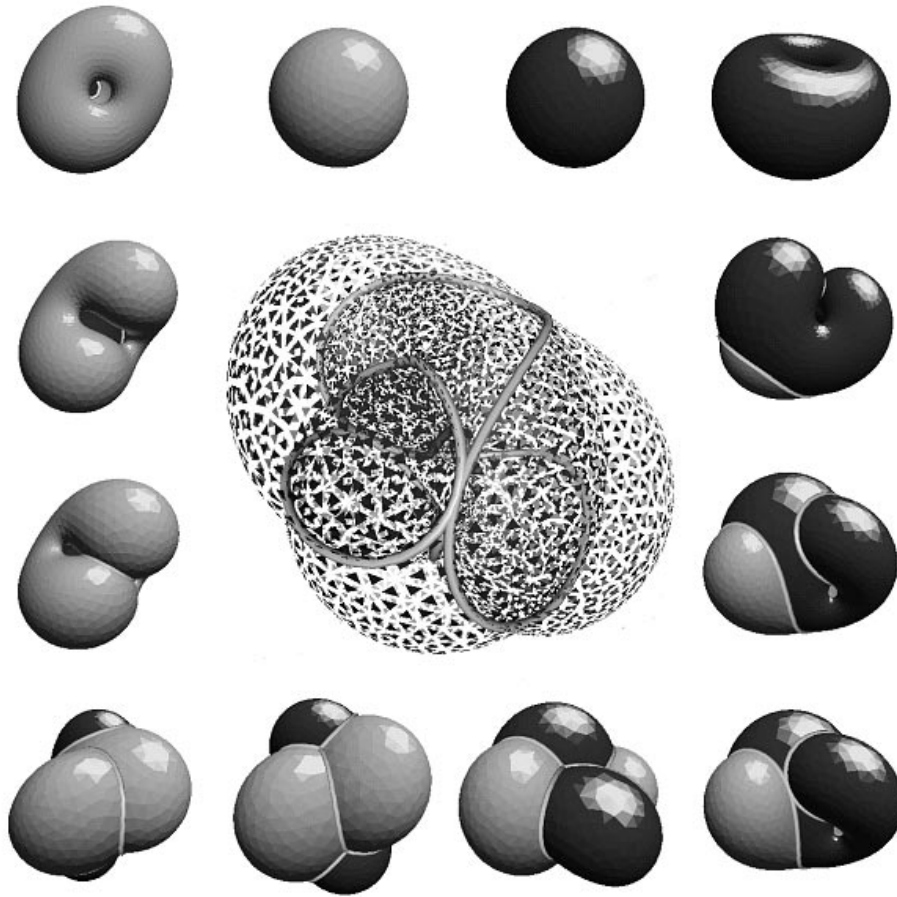


FIGURE 9. This is the same eversion shown in Fig. 8, but rendered with solid surfaces. Again, we start at the top with a round sphere, and proceed clockwise. Down the right-hand side we see the creation first of two double-curves, and then of a pair of triple points. (Another pair is created at the same time in back; the eversion always has two-fold rotational symmetry.) Across the bottom, we go through the halfway-model, interchanging the roles of the dark and light sides of the surface. Up the left column, we see the double-curves disappear one after the other. (This time, the figures on the left are not exactly the same stages as the corresponding ones on the right.) In the center, we examine the double locus just when pairs of triple points are being created, by shrinking each triangle of the surface to a quarter of its normal size.

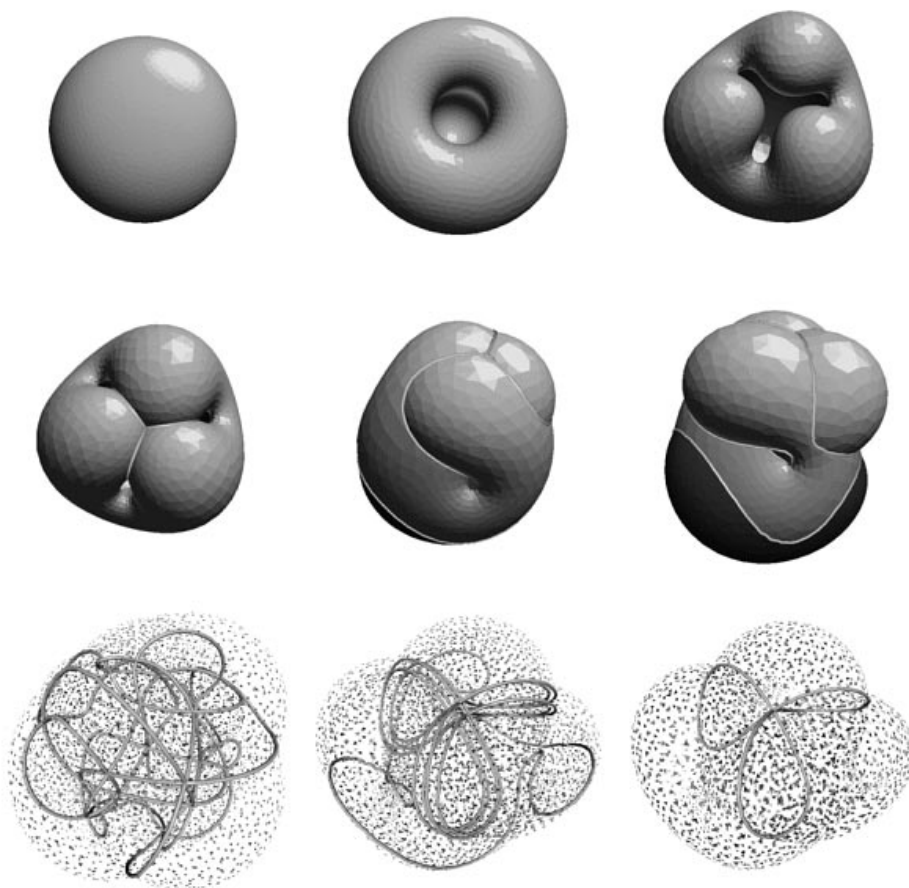


FIGURE 10. This three-fold minimax eversion starts (top row, left-to-right) with a gastrula stage like that of the two-fold eversion, but the three-fold symmetry means that three fingers reach up from the neck instead of two. They intersect each other (middle row, left-to-right) and then twist around, while complicated things are happening inside. The bottom left image is at the same stage as the middle right image, but with triangles shrunk to show the elaborate double-curves. These double-curves separate into two pieces (bottom row, left-to-right), one of which is a four-fold cover of the propeller-shaped double-curve seen in the Boy's surface halfway-model (lower right).

## REFERENCES

- [AS] Frederick J. Almgren, Jr. and John M. Sullivan. Visualization of soap bubble geometries. *Leonardo* **24**(1992), 267–271. Reprinted in *The Visual Mind*.
- [Apé] François Apéry. An algebraic halfway model for the eversion of the sphere. *Tohoku Math. J.* **44**(1992), 103–150. With an appendix by Bernard Morin.
- [BM] Thomas Banchoff and Nelson L. Max. Every sphere eversion has a quadruple point. In D. N. Clark, G. Pecelli, and R. Sacksteder, editors, *Contributions to Analysis and Geometry*, pages 191–209. Johns Hopkins Univ. Press, 1981.
- [Bra] Kenneth A. Brakke. The Surface Evolver. *Experimental Math.* **1**(1992), 141–165.
- [BS] Kenneth A. Brakke and John M. Sullivan. Using symmetry features of the surface evolver to study foams. In Hege and Polthier, editors, *Visualization and Mathematics*, pages 95–117. Springer, 1997.
- [Bry] Robert Bryant. A duality theorem for Willmore surfaces. *J. Differential Geometry* **20**(1984), 23–53.
- [Fra] George Francis. *A Topological Picturebook*. Springer, New York, 1987.
- [FM] George Francis and Bernard Morin. Arnold Shapiro’s eversion of the sphere. *Math. Intelligencer* **2**(1979), 200–203.
- [FSH] George Francis, John M. Sullivan, and Chris Hartman. Computing sphere everisions. In Hege and Polthier, editors, *Mathematical Visualization*, pages 237–255. Springer, 1998.
- [FSK<sup>+</sup>] George Francis, John M. Sullivan, Robert B. Kusner, Kenneth A. Brakke, Chris Hartman, and Glenn Chappell. The minimax sphere eversion. In Hege and Polthier, editors, *Visualization and Mathematics*, pages 3–20. Springer, 1997.
- [HKS] Lucas Hsu, Rob Kusner, and John M. Sullivan. Minimizing the squared mean curvature integral . . . . *Experimental Math.* **1**(1992), 191–207.
- [Hug] John F. Hughes. Another proof that every eversion of the sphere has a quadruple point. *Amer. J. Math.* **107**(1985), 501–505.
- [Kus] Rob Kusner. Conformal geometry and complete minimal surfaces. *Bull. Amer. Math. Soc.* **17**(1987), 291–295.
- [Lev] Silvio Levy. *Making Waves: A Guide to the Ideas Behind Outside In*. AK Peters, Wellesley, MA, 1995.
- [LMM] Silvio Levy, Delle Maxwell, and Tamara Munzner. *Outside In*. AK Peters, Wellesley, MA, 1994. Narrated videotape (21 min) from the Geometry Center.
- [LY] Peter Li and S. T. Yau. A new conformal invariant and its applications to the Willmore conjecture . . . . *Invent. Math.* **69**(1982), 269–291.
- [Max] Nelson L. Max. *Turning a Sphere Inside Out*. International Film Bureau, Chicago, 1977. Narrated videotape (21 min).
- [MP] Bernard Morin and Jean-Pierre Petit. Le retournement de la sphère. In *Les Progrès des Mathématiques*, pages 32–45. Pour la Science/Belin, Paris, 1980.
- [Pet] Ivars Peterson. Contemplating the Optiverse: Surreal films. *Science News* **154**(October 1998), 232–234.
- [Phi] Anthony Phillips. Turning a sphere inside out. *Sci. Amer.* **214**(1966), 112–120.
- [Sma] Steven Smale. A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.* **90**(1959), 281–290.
- [SFL] John M. Sullivan, George Francis, and Stuart Levy. The Optiverse. In Hege and Polthier, editors, *VideoMath Festival at ICM’98*. Springer, 1998. Narrated videotape (7 min).
- [Wil] Thomas J. Willmore. A survey on Willmore immersions. In *Geometry and Topology of Submanifolds, IV (Leuven, 1991)*, pages 11–16. World Sci. Pub., 1992.

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