The order of numbers in the Second Viennese School of Music

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Mathematics and Music seem nowadays independent areas of knowledge. Nevertheless, strong connections exist between them since ancient times. Twentieth-century music is no exception, since in many aspects it admits an obvious mathematical formalization.

In this article some twelve-tone music rules, as created by Schoenberg, are presented and translated into mathematics. The representation obtained is used as a tool in the analysis of some compositions by Schoenberg, Berg, Webern (the Second Viennese School) and also by Milton Babbitt (a contemporary composer born in 1916).

The Second Viennese School of Music

> *The 12-tone music was condemned by the Nazis (and forbidden in the occupied Europe) because its author was a Jew; by the Stalinists for having a bourgeois cosmopolitan formalism; by the public for being different from everything else.*
>
> Roland de Candé [2]

Arnold Schoenberg (1874-1951) was born in Vienna, in a Jewish family. He lived several years in Vienna and in Berlin. In 1933, forced to leave the Academy of Arts of Berlin, he moved to Paris and short after to the United States. He lived in Los Angeles from 1934 until the end of his life.

In 1923 Schoenberg presented the *twelve-tone music* together with a new composition method, also adopted by Alban Berg (1885-1935) and Anton Webern (1883-1945), his students since 1904.

The three composers were so closely associated that they became known as the *Second Viennese School of Music.*

The Twelve-Tone Method

Tonality consists of the relations, melodic and harmonic, between the several notes of a given scale. In tonal music, the most important note is the so called *tonic*: around the tonic gravitates both melody and harmony.

Schoenberg wanted to remove the prevailing role of the tonic, as well as the hierarchy that tonality imposes between the seven notes of the traditional scale. With this idea in mind, in 1923 he established the *Twelve-Tone Method*. This method proposes to give the same merit to each note of the chromatic scale; in a 12-tone composition, all chromatic notes appear exactly the same number of times. The basis for a 12-tone composition is a sequence of the 12 distinct music pitches (without repetitions), appearing in any octave and combined under any rhythm. This sequence is called *12-tone series* or *12-tone row.*

The basis of each composition is one single series; only this basic series or some others related with this one by symmetry can be used in the composition. The composer can use the series in its *original* form, or with its intervals *inverted*, or backwards (*retrograde*), or *transposed* by some half-tones. Imposing as a rule that no series can start before the previous one is finished, at the end of the composition all 12 notes have in fact appeared the same amount of times.

The first series of History of Music is the one used in the fifth piece of *Five Piano Pieces, Opus 23*, from Schoenberg, written in 1923: C$^\sharp$, A, B, G, A$^\flat$, G$^\flat$, Si$^\flat$, D, E, E$^\flat$, C, F.

Mathematical Formalization of Twelve-tone Music

People accuse me of being a mathematician, but I am not a mathematician, I am a geometer.

Arnold Schoenberg

In this section we use numbers to represent musical notes. Recall that in a 12-tone row, notes with the same name are considered equivalent, even if they belong to different octaves. We start by identifying consecutive notes with consecutive integers. In this way, if C is represented by the integer 1, then C$^\sharp$ is represented by 2, B by 12 and the following C has to be again represented by 1. The difference between two integers representing two notes gives the number of half-steps of the interval between the two corresponding notes (ignoring octaves). For instance, C and G can be represented by 1 and 8, respectively, and they define an interval of a perfect fifth, or 7 half-tones (or an interval of 7+12 half-tones, or $7+12k$ half-tones, for some integer k).

Given an integer p, define the set $[p] = \{p + k \times 12, \text{ for some integer } k\}$. Notice that $[1] = [49] = [-11]$, as well as $[5] = [29] = [-7]$. The set $\mathbb{Z}_{12} = \{[0], [1], [2], \ldots, [11]\}$ is called *set of integers modulo 12* and the set $[p]$ is called *equivalence class of p modulo 12.* Addition in $\mathbb{Z}_{12}$ (*addition modulo 12)* is defined as the usual addition in $\mathbb{Z}$, having into account that numbers differing by a multiple of 12 are equivalent. For instance, we have $(3+8)mod_{12} = 11$, $(5+7)mod_{12} = 0$ and $(7+7)mod_{12} = 2$. Using the equivalence between numbers differing by a multiple of 12, we define *symmetric of an integer modulo 12* obtaining, for instance, $(-5)mod_{12} = 7$, $(0)mod_{12} = 0$, $(-11)mod_{12} = 1$.

We may now define *12-tone series* as a permutation of the integers $0, 1, 2, \ldots, 11$, recalling that each integer has to be seen as an equivalence class modulo 12.

Example 1

In Schoenberg's *Piano Concerto, Opus 42* (1942), the basic row is

E$^\flat$, B$^\flat$, D, F, E, C, F$^\sharp$, A$^\flat$, D$^\flat$, A, B, G.

Considering E$^\flat \equiv 0$, E $\equiv 1$, and so forth, this series, which we represent by S, can be re-written as

$$S = (0,\ 7,\ 11,\ 2,\ 1,\ 9,\ 3,\ 5,\ 10,\ 6,\ 8,\ 4). \tag{1}$$

Series related to a given series

In order to collect all material to be used in a musical composition, after choosing the basic series (or basic row), it is necessary to list all series which can be obtained by symmetry from the basic one.

Suppose that the *basic row* is defined by

$$P = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}) \tag{2}$$

The *retrograde* row, which we represent by $R(P)$, is obtained from P playing it backwards

$$R(P) = (a_{12}, a_{11}, a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1) \tag{3}$$

The *inverse* row, which we represent by $I(P)$, is obtained from the basic row P maintaining its first note and inverting all intervals between consecutive notes, in such a way that an ascending interval becomes descendent and vice-versa.

Denoting the entries of $I(P)$ by a_k^*, $k = 1, \ldots, 12$, they verify the following

$$\begin{array}{rcl} a_1^* & = & a_1 \\ a_2^* & = & a_1^* - (a_2 - a_1) \\ a_3^* & = & a_2^* - (a_3 - a_2) \\ & \vdots & \\ a_{12}^* & = & a_{11}^* - (a_{12} - a_{11}) \end{array}$$

If $a_1 = 0$ in the basic row, then the inverse row $I(P)$ takes the form

$$I(P) = ((-a_1)mod_{12}, \ldots, (-a_{12})mod_{12}), \tag{4}$$

that is, the *inverse* series $I(P)$ of P is represented by the substitution of each entry of the series by its symmetric with respect to addition in $\mathbb{Z}_{12}$ (property only valid if we associate to the first note of the basic series the number 0).

The *retrograde inverse* row, which we represent by $RI(P)$, is obtained from P by applying the previous two operations. In case $a_1 = 0$ in P, the row $RI(P)$ is

$$RI(P) = ((-a_{12})mod_{12}, \ldots, (-a_1)mod_{12}). \tag{5}$$

Any *transposition by k half-tones of the basic row* is obtained from P by adding k (modulo 12) to each entry of P. The transposition of P by k half-tones is

$$P_k = ((a_1 + k)mod_{12}, ..., (a_{12} + k)mod_{12}) \tag{6}$$

In a similar way we obtain the transpositions by k half-tones of the retrograde row, the inverse row and the retrograde inverse row.

The Matrix of Series

The rows related with the basic row P can be organized in one matrix $M(P)$, as follows.

Let P be the series

$$P = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12})$$

with $a_1 = 0$.

Let $M(P)$ be the 12×12 matrix obtained in the following way: the 1^{st} row of $M(P)$ is the basic series P; the 1^{st} column of $M(P)$ is the inverse series $I(P)$; since $a_1 = 0$, the entries of the 1^{st} column are given by (4). The remaining rows are transpositions of P, being that transposition by k half-tones (P_k) if k is the 1^{st} entry of the row. In the same way, the remaining columns are transpositions by k half-tones of the inverse row (I_k) (where k is the 1^{st} entry of the column), the 1^{st} row backwards is the retrograde ($R(P)$) of the basic row, the 1^{st} columm up side down is the retrograde inverse ($RI(P)$), the remaining rows backwards are transpositions of the retrograde row (R_k) and the remaining columns up side down are the transposition of the retrograde inverse (RI_k).

The series related with the basic series are at most 48, being less than 48 if there are some symmetries in the basic series, as we shall see.

Example 2

The matrix of series for the Schoenberg's *Piano Concerto, Opus 42* (see Example 1) is as follows.

Figure 1: The 1^{st} row contains the elements $a_1, \dots, a_{12}$; the 1^{st} column contains the elements $-a_1, \dots, -a_{12}$. Since each row i is obtained from the first row by the addition of $-a_i$ (which is the 1^{st} entry of the row i), the element of the matrix in the position (i, j) (row i, column j) is equal to $a_j - a_i$. Obviously, the main diagonal of the matrix $M(S)$ is composed by zeros.

Hexachords

If we consider a series just a permutation of the 12 integers $0, 1, 2, \dots, 11$, then the total number of possible series is $\mathcal{P}(12) = 12! = 479.001.600$. Of course not all permutations can be used as series, since composers are certainly interested in series with special musical properties.

Example 3

Consider the row P

$$P = (0, 2, 7, 5, 10, 9, 3, 4, 11, 1, 8, 6)$$

Transposing P by 6 half-tones, we obtain

$$P_6 = (6, 8, 1, 11, 4, 3, 9, 10, 5, 7, 2, 0)$$

The retrograde of P coincides with the transposition of P by 6 half-tones, that is, $P_6 = R(P)$. Obviously the row P has less than 48 different associated rows.

Proposition: *Let P be a given row. If $R(P) = P_k$, then k=6.*

Proof

Suppose that the retrograde of the series P

$$P = (a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6, ...,\ a_{12})$$

is equal to

$$P_k = (a_1 + k,\ a_2 + k,\ a_3 + k, \ldots,\ a_{12} + k),$$

for some $k \in \{0, 1, 2, \ldots, 11\}$. Then $(a_1 + k = a_{12}) mod_{12}$ and $(a_{12} + k = a_1) mod_{12}$. From $(a_1 = a_1 + 2k) mod_{12}$ we conclude $k = 6$. ∎

Quite often, the choice of a particular series such as the one in Example 3 needs the study of some sub-sets of $\mathbb{Z}_{12}$. We give particular attention to *hexachords*, sub-sets of $\mathbb{Z}_{12}$ containing six elements. The series P defined in (2) can be divided in two hexachords as follows

$H_1(P)$	$H_2(P)$
$a_1, a_2, a_3, a_4, a_5, a_6$	$a_7, a_8, a_9, a_{10}, a_{11}, a_{12}$

Notice now that in order to have $R(P) \equiv P_6$, neither $H_1(P)$ nor $H_2(P)$ can contain two elements of $\mathbb{Z}_{12}$ with a difference (module 12) equal to 6. Furthermore, in order to have $R(P) = P_6$, $H_2(P)$ has to be equal to the retrograde of $H_1(P)$, transposed by 6 half-tones.

Proposition: *Divide the series P in two hexachords $P = (H_1(P), H_2(P))$.*
P verifies the condition $R(P) = P_6$ if and only if
(a) $H_1(P)$ does not contain any pair of notes differing by 6 half-tones;
(b) $H_2(P) = R(H_1(P) + 6)$.

The amount of series with the property $R(P) = P_6$ is equal to $12 \times 10 \times 8 \times 6 \times 4 \times 2 = 46\,080$.

Definition: Given two series A and B we say that *A and B combine* if both first hexachords together contain all 12 notes, without repetitions.

Let A and B be the series

	1^{st} hexachord	2^{nd} hexachord
A	$H_1(A) : a_1, a_2, a_3, a_4, a_5, a_6$	$H_2(A) : a_7, a_8, a_9, a_{10}, a_{11}, a_{12}$
B	$H_1(B) : b_1, b_2, b_3, b_4, b_5, b_6$	$H_2(B) : b_7, b_8, b_9, b_{10}, b_{11}, b_{12}$

When A and B combine, both sequences $(H_1(A), H_2(B))$ and $(H_1(B), H_2(A))$ also define series.

Obviously, any row combines with its own retrograde. This trivial property has been cleverly used by Webern, as shown in Example 4.

Example 4

In Webern's *Piano Variations, Opus 27,* the basic row is

$$P : (0, 1, 9, 11, 8, 10, 4, 5, 6, 2, 3, 7)$$

with E≡0.

Figure 2: *Webern's Piano Variations Op. 27.*

We separate the row P in two hexachords.

$H_1(P)$	$H_2(P)$
$(0, 1, 9, 11, 8, 10)$	$(4, 5, 6, 2, 3, 7)$

The right hand starts playing the 1^{st} hexachord of P followed by the 2^{nd} hexachord of the retrograde $R(P)$, while the left hand plays the 1^{st} hexachord of $R(P)$, followed by the 2^{nd} hexachord of P, leading to the following palindromic structure

right hand:	0	1	9	11	8	10	10	8	11	9	1	0
left hand:	7	3	2	6	5	4	4	5	6	2	3	7

as can easily be seen in Figure 2.

Example 5

In Schoenberg's *Piano Piece, Opus 33a,* the basic series is

$$P = (0, 7, 2, 1, 11, 8, 3, 5, 9, 10, 4, 6)$$

with $\mathrm{B}^\flat \equiv 0$.

Figure 3: *Schoenberg's Piano Piece, Op. 33a.* When both right and left hands have only played the first hexachord of the corresponding series, (P_0 for the right hand and I_5 for the left hand), already both hands together have played all 12 notes of the chromatic scale.

We notice first that P combines with the transposition by 5 half-tones of its inverse $I(P)$.

The first hexachord H_1 of the series P is

$$H_1 = (a_i)_{1 \leq i \leq 6} = (0, 7, 2, 1, 11, 8);$$

the inverse of H_1 is

$$I(H_1) = (-a_i)_{1 \leq i \leq 6} = (0, 5, 10, 11, 1, 4);$$

the transposition by five half-tones of the inverse of H_1 is

$$I_5(H_1) = (-a_i + 5)_{1 \leq i \leq 6} = (5, 10, 3, 4, 6, 9)$$

and $I_5(H_1)$ is indeed a complementary hexachord of H_1.

Creating 12-tone music

> *I can tell you, dearest friend, that if became known how much friendship, love and a world of human and spiritual references I have smuggled into these movements, the adherents of programme music (should there be any left) would go mad with joy.*

Berg, in a letter to Schoenberg, about the *Chamber Concerto*, 1925.

In this section we try to show, by means of some musical examples, that despite their seemingly strict rules, the character of the composer can still be recognized in a 12-tone music composition.

Example 6
The music of Alban Berg is intensely expressive, almost romantic. In fact, Berg was able to compose 12-tone music without being entirely away from tonality. The work of Berg can be seen as a compromise between traditional principles (tonality) and innovative principles (atonality, 12-tone method), fact that indeed is quite evident in his Violin Concerto.

In Berg's *Violin Concerto*, 1935, the basic row is

$$P = (0, 3, 7, 11, 2, 5, 9, 1, 4, 6, 8, 10)$$

with G≡0. Exceptionally, and to make clear the tonal character of this piece, we present this series using the name of the notes involved.

	G	
G minor	$B^\flat$	
	D	
	$F^\sharp$	D major
	A	
A minor	C	
	E	
	$G^\sharp$	E major
	B	
	$C^\sharp$	
	$D^\sharp$	
	F	

The basic row crosses several tonalities: the first three notes define the tonality of G minor, moving successively to the tonalities of D major, A minor, E major, finishing with four notes presenting a sequence of three whole tones.

Figure 4: Violin Concerto, Berg, 1935.

Example 7

Anton Webern is, from the Vienna Trio, the one arriving further away from the tonal system. The compositions by Webern are in general brief and dry, while the traditional melodic line is substituted by individual notes, with no melodic connection between them, some times even confined to different instruments, as it is evident in his *Concerto for Nine Instruments, Op. 24.*

The basic series for this concert was carefully chosen in order to have an internal structure with interesting symmetries. The basic series is

$$P = (0, 11, 3; 4, 8, 7; 9, 5, 6; 1, 2, 10),$$

with B≡0. We consider the first three notes of the series, $0, 11, 3$, and deal with this set of notes as a "mini-row" which we represent by T. We have

$$\begin{array}{rl} T: & (0, 11, 3) \\ I(T): & (0, 1, 9) \\ R(T): & (3, 11, 0) \\ RI(T): & (9, 1, 0) \end{array}$$

Notice further that

$$\begin{array}{rl} I(T) + 1: & (1, 2, 10) \\ R(T) + 6: & (9, 5, 6) \\ RI(T) + 7: & (4, 8, 7) \end{array}$$

The basic row P can in fact be completely re-written in terms of the mini-row T, as follows

$$P = (T, RI_7(T), R_6(T), I_1(T)).$$

Figure 5: Concerto for Nine Instruments, Op. 24. The series is separated in four small groups of three notes each, which become even more independent since they are distributed by different instruments and played with different speed.

Example 8

After Schoenberg, Berg and Webern, other composers have used the idea of a series in musical elements other than notes. Metric, intensity of sound, rhythm, timbre, among others, have been incorporated into serial structures. In Example 8, we see how Milton Babbitt (American composer born in 1916) uses the idea of basic series to also establish the duration and intensity of sound.

Figure 6: Three Compositions for piano, No.1, Babbitt (1947)

The series of notes used in this piece is

$$P = (0, 5, 7, 4, 2, 3, 9, 1, 8, 11, 10, 6)$$

with $0 \equiv \mathrm{B}^\flat$.

The transposed of P by 6 half-tones is

$$P_6 = (6, 11, 1, 10, 8, 9, 3, 7, 2, 5, 4, 0).$$

Notice that P combines with its transposition P_6. On the other hand, the retrograde of P is

$$R(P) = (6, 10, 11, 8, 1, 9, 3, 2, 4, 7, 5, 0),$$

the inverse of P is

$$I(P) = (0, 7, 5, 8, 10, 9, 3, 11, 4, 1, 2, 6)$$

and the retrograde inverse of P is

$$RI(P) = (6, 2, 1, 4, 11, 3, 9, 10, 8, 5, 7, 0),$$

obtaining

$$RI_1(P) = (7, 3, 2, 5, 0, 4, 10, 11, 9, 6, 8, 1).$$

Notice that $R(P)$ combines with $RI_1(P)$.

The structure of the first four measures is as follows

measure	I	II	III	IV
right hand	$H_1(P_6)$	$H_2(P_6)$	$H_1(R)$	$H_2(R)$
left hand	$H_1(P)$	$H_2(P)$	$H_1(RI_1)$	$H_2(RI_1)$

Because P combines with P_6 and $R(P)$ combines with $RI(P)_1$, the notes of the two hexachords played by both left and right hands in each measure are all the 12 notes of the scale.

For the duration of sounds, a sequence of four numbers is used,

$$D = (5, 1, 4, 2),$$

with operations modulo 6 (in a similar way as $\mathbb{Z}_{12}$, we define the set of *integers modulo 6*, $\mathbb{Z}_6$, with operations defined in such a way that integers differing by a multiple of 6 are equivalent). Each entry of the series D indicates the quantity of consecutive sixteenths appearing in each group. More exactly, the fact that the initial entry is equal to 5 indicates that the first five notes of the composition must form a group of five consecutive and linked sixteenths. Obviously, the addition of all four entries of D is equal to 12, the number of notes of the series.

Notice further that

$$\begin{array}{ll} D: & 5,\ 1,\ 4,\ 2 \\ I(D): & 1,\ 5,\ 2,\ 4 \\ R(D): & 2,\ 4,\ 1,\ 5 \\ RI(D): & 4,\ 2,\ 5,\ 1 \end{array}$$

Any time the row P (or a transposition of P) is played, the rhythm indicated by D is used; when the inverse of P (or a transposition of $I(P)$) is played, the duration of the notes has to respect $I(D)$; retrogrades are played with rhythm given by $R(D)$, and $RI(D)$ indicates the rhythm of the retrograde inverse of P (or transpositions of $RI(P)$).

There is also a correspondence between the form of the series and the intensity of the sound: the row P (or a transposition of P) is always played *mezzo piano (mp)*; $I(P)$ (or a transposition of $I(P)$) is played *forte (f)*; retrogrades are played *mezzo forte (mf)* while $RI(D)$ are played *piano (p)*.

notes	rhythm	intensity
P	D	*mezzo piano (mp)*
$R(P)$	$R(D)$	*mezzo forte (mf)*
$I(P)$	$I(D)$	*forte (f)*
$RI(P)$	$RI(D)$	*piano (p)*

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