A mathematical interpretation of expressive intonation

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1. Introduction

In their talks, Erich Neuwirth and Carlotta Simens have given a description of diverse musical temperaments and, respectively, of some uses of equal temperament.

Presently I will describe a mathematical frame in which an old musical practice known as "expressive intonation" by singers and string players can find an explanation which would be impossible in any system of equal temperament. This does not mean that a phenomenon of the sort cannot occur in piano playing, but in this case it belongs to the concept of homonymy : each black key being a sort of "double entendre" as it is in Beethoven's Moonlight Sonata where the black key which is just above C is considered to mean $C^{\#}$ in the first and last movements and to mean D^b in the second.

We will show how players who are not slaves to a fixed pitch, may dispel this ambiguity.

2. "Petite fleur"

I have chosen to let the audience listen to a well-known piece of popular music as an example of this practice : this is "Petite Fleur" played by S. Bechet on an alto saxophone.

The theme of the refrain begins by the top E^b of the minor sixth $\frac{E^b}{G}$. According to all treatises this minor sixth is attracted to the perfect fifth $\frac{D}{G}$. The musicologist D. Cooke says in [C] that the expectation of the perfect fifth in this context creates a sentiment of anguish, and this can be checked if you pay attention to the meaning of the words underneath the notes of "Petite Fleur".

Each time this refrain reappears, S. Bechet tries to emphasize in his way of playing this sentiment of expectancy. Among other things, the listener notices that his E^b is intentionally flat : too flat according to equal temperament but nevertheless marvellously in tune !

H. Helmholtz ([Hh], p. 428) was perfectly aware of this shortcoming of equal temperament, when he wrote :

"When the organ took the lead among musical instruments it was not yet tempered. And the pianoforte is doubtless a very useful instrument for making the acquaintance of musical literature, or for domestic amusement, or for accompanying singers. But for artistic purposes its importance is not such as to require its mechanism to be made the basis of the whole music system".

"I think that many of our best musical performances owe their beauty to an unconscious introduction of the natural system, and that we should oftener enjoy their charms if that system were taught pedagogically, and made the foundation of all instruction in music, in place of the tempered intonation which endeavours to prevent the human voice and bowed instruments from developing their full harmoniousness, for the sake of not interfering with the convenience of performers on the pianoforte and the organ".

3. Scale constructions

The tuning of string instruments is usually based on the pure fifth, which is the interval $\frac{3}{2}$. The reason for this is that it is very easy to produce experimentally, as you only have to check that the harmonic of order 2 of the upper string concides with the harmonic of order 3 of the lower string.

If we extend this construction mentally in both directions, we find the fractions of a subgroup of $\mathbb{Q}_+^*$ generated by $\frac{3}{2}$, subgroup usually noted $<\frac{3}{2}>$ by mathematicians.

If we translate these intervals in the first octave (the interval $[1,2[$) modulo the powers of 2 (namely the elements of the group $<2>$) we can notice that, although there are near concidences, all the fractions we get are different.

Expressed in other terms the "equation of the musicians" namely :

$$\left(\frac{3}{2}\right)^x = 2^y$$

with $(x,y) \in \mathbb{Z}^2$, is impossible except for the trivial solution $(x,y) = (0,0)$.

This was well known to Pythagoras, but at the same time Pythagoras noticed that $(x,y) = (12,7)$ was a very good approximate solution since $\frac{3^{12}}{2^{19}}$ is very near 1. We will call this quantity "Pythagoras' comma" and denote it by ϖ. As D. Cooke puts it in $[C]$ p. 44 :

"We may say that whereas musically we want the equation $\frac{3^{12}}{2^{19}} = 1$, the correct mathematical equation is $\frac{3^{12}}{2^{19}} = \left(\frac{3}{2}\right)^{12} \times \frac{1}{2^7} = 1,013642\ldots$".

Solving "mathematically" this impossible equation can be done in several ways :

1) **the official solution**, which consists in taking "2" = 2 and "$\frac{3}{2}$" $= 2^{\frac{7}{12}} \in < r_0 >$ with $r_0 = 2^{\frac{1}{12}} \in \mathbb{R}_+^*$.

This can be given a rigourous mathematical meaning if you consider the homorphism h_0 of the free abelian group $<2,3>$ on $<r_0>$ given by :

$$h_0(2^x 3^y) := r_0^{12x+19y} \in < r_0 >$$

The theorem of isomorphism tells us then that :

$$\mathrm{Im}(h_0) = < r_0 > \cong < 2,3 > / Ker(h_0)$$

with $\mathrm{Ker}(h_0) = < \frac{3^{12}}{2^{19}} > = < \varpi >$.

We will say that $< r_0 >$ is the official tempered scale.

2) **the solution of S. Cordier** $[Cr]$ which is similarly constructed when you consider h_c given by :

$$h_c(2^x 3^y) := r_c^{12x+19y} \in < r_c >$$

with $r_c := \left(\frac{3}{2}\right)^{1/7} \in \mathbb{R}_+^*$.

As above we deduce that :

$$\mathrm{Im}(h_c) = < r_c > \cong < 2,3 > / \mathrm{Ker}(h_c)$$

with $\mathrm{Ker}(h_c) = < \frac{3^{12}}{2^{19}} > = < \varpi >$.

We will say that $< r_c >$ is S. Cordier's tempered scale.

3) The above isomorphisms suggest to consider an "**abstract scale**" which is :

$$< 2,3 > / < \varpi >$$

when ϖ denotes Pythagoras' comma $\dfrac{3^{12}}{2^{19}}$. From what we have already said we know that this quotient group is isomorphic to $\mathbb{Z}$ and we can check that it is generated by the class of $\dfrac{2^8}{3^5}$. In fact it can be verified that

$$h\left(\frac{2^8}{3^5}\right)^{12} = h(2)$$

where h means either h_0 or h_c, and this means that the twelfth power of the class of $\dfrac{2^8}{3^5}$ is the class of the octave.

These solutions can be summed up in a same diagram :

$$N \hookrightarrow G \underset{\varphi}{\overset{s}{\leftrightarrows}} \mathbb{Z}$$

where G means $< r >$ in the case of the tempered scales and $< 2, 3 >$ is the case of the abstract scale, where N means $< 1 >$ in the case of tempered scales and $< \varpi >$ in the case of the abstract scale, where φ is either $r^n \longmapsto n$ or $2^x 3^y \longmapsto 12x + 19y$ and where s is a section chosen such that $\varphi \circ s = \mathrm{id}_{\mathbb{Z}}$. Mathematicians say that G is an extension of N by $\mathbb{Z}$. In both cases the extensions are "trivial" in the sense that we can take for s a homomorphism (the extension is "split").

But, in the second case, musicians do not take for s a homomorphism !

L. Euler helps us to understand the musical choice we have to make by the following remark :

"The sense of hearing is accustomed to identify with a single ratio, all the ratios which are only slightly different from it, so that the difference between them be almost imperceptible".

By "difference" Euler naturally means "interval" or equivalently the quotient of the two ratios.

So, in the case of the abstract scale, we decide to take for $s(\varphi(x))$ the "simplest" ratio ρ in the class of x modulo N.

But what meaning shall we attach to this concept of simplicity ?

We will chose ρ as close as possible to 1 for the "harmonic distance" on $\mathbb{Q}_+^*$ (see [He], §1 for the definition of this distance) or, equivalently, we will take $\rho = \dfrac{n}{d}$ (the fraction being reduced in its simplest terms) with $\sup(n, d)$ minimal in the class of x modulo N.

It turns out that this choice of ρ is unambiguous in the sense that there is only **one** element in xN which satifies our condition.

The first twelve values of $s(n)$ are :

n	0	1	2	3	4	5	6	7	8	9	10	11
$s(n)$	1	$\frac{2^8}{3^5}$	$\frac{3^2}{2^3}$	$\frac{2^5}{3^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$	$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{2^7}{3^4}$	$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$	$\frac{3^5}{2^7}$
name	C		D		E	F		G		A		B
		D^b		E^b			$F\#$		A^b		B^b	

and we can check that the intervals of the C major scale are exactly the ones given by E. Neuwirth in his empirical construction.

Remarks

1) As we noticed in the introduction there is an ambiguity in the naming of the notes which do not belong to the C major scale. This is the reason why we placed their names on a lower line. We will go back to this later.

2) This construction is the paradigm of an infinite series of Pythagorean chromatic scales which have respectively :

$$1,\ 2,\ 5,\ 12,\ 41,\ 53,\ \text{etc.}$$

degrees in one octave.

The usual Pythagorean scale is the fourth in this series and the next one (with 41 degrees in an octave) is called Janko's scale.

It is given by the diagram :

$$\underset{=<\frac{2^{65}}{3^{41}}>}{N} \hookrightarrow \underset{=<2,3>}{G} \underset{\varphi}{\overset{s}{\leftrightarrows}} \mathbb{Z}$$

and the generator of G/N is the class of $\varpi = \dfrac{3^{12}}{2^{19}}$.

The "concrete scale" $s \circ \varphi(G)$ begins as follows :

0	1	2	3	4	5	6	7
1	$\frac{3^{12}}{2^{19}}$	$\frac{2^{27}}{3^{17}}$	$\frac{2^{8}}{3^{5}}$	$\frac{3^{7}}{2^{11}}$	$\frac{3^{19}}{2^{30}}$	$\frac{2^{16}}{3^{10}}$	$\frac{3^{2}}{2^{3}}$
C			D^b	$C\#$			D

So, in Janko's scale, the Pythagorean diatonic semitone $\dfrac{D^b}{C}$ is $\dfrac{3}{7}$ of the whole Pythagorean tone $\dfrac{D}{C}$ and the chromatic semitone $\dfrac{C\#}{C}$ is $\dfrac{4}{7}$ of the whole tone. But we will come back later to the definition of those semitones.

3) The sixth scale is Mercator's scale.
It is given by the diagram :

$$\underset{=<\frac{2^{65}}{3^{41}}>}{N} \hookrightarrow \underset{=<2,3>}{G} \underset{\varphi}{\overset{s}{\leftrightarrows}} \mathbb{Z}$$

The generator of the "abstract" scale G/N is the class of $\dfrac{2^{65}}{3^{41}}$.

One can check that the Pythagorean whole tone $\dfrac{3^2}{2^3}$ is divided in nine parts and that the diatonic semi-tone is $\dfrac{4}{9}$ of a whole tone while the chromatic semi-tone is $\dfrac{5}{9}$ of a whole tone.

4) Finally if we consider the ratio of the logarithms of the chromatic semi-tone and of the whole tone we find : $0,4425\ldots$ and it can be proved that it is the limit of the number of degrees contained in those intervals when the scale tends to infinity in the Pythagoric series referred above.

As a comparison we have :

$$\frac{3}{7} = 0,4285\ldots < \frac{\log C^{\#}}{\log C} = 0,4425\ldots < \frac{4}{9} = 0,4444\ldots$$

So the theories about Holder's comma seem consistent with this construction.

4) Playing "Petite Fleur" in the Pythagorean scale

The score of "Petite Fleur" shows that the first note is an E^b and that the next chord contains D and $F^{\#}$, so the abstract scale in which it is written is G minor, and this has to be present in the mind of all the listeners from the beginning.

A player like S. Bechet (I mean a good player) will play the E^b in a way which will make it quite different from a $D^{\#}$ (to suppress the homonymy) and the $F^{\#}$ in a way which will make it quite different from a G^b : this can be done in the Pythagorean scale but not in a tempered scale !

And in so doing this player will follow the teaching of masters like P. Casals [B] or the findings of musicologists like D. Cooke [C].

5) Mathematical interpretation of expressive intonation

Sharps and flats are usually introduced by transposing by successive fifths the C major scale. When you want to introduce sharps and flats in the Pythagorean scale you must therefore multiply the ratio by $\frac{3}{2}$ several times (translation).

In fact the practice of transposition of the scale of C major was very popular in the XIX^{th} century, when it was taught systematically by the Tonic Sol-faists (see [Hh] p. 422-428) as the basis for natural intonation. Let us introduce $F^{\#}, C^{\#}$ and B^b.

	C	$C^{\#}$	D	E	F	$F^{\#}$	G	A	B^b	B	C
C maj	1		$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$		$\frac{3}{2}$	$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$	2
G maj	1		$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$		$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$	2
F maj	1		$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$	$\frac{2^2}{3}$		$\frac{3}{2}$	$\frac{3^3}{2^4}$	$\frac{2^4}{3^2}$		2
D maj		$\frac{3^7}{2^{11}}$	$\frac{3^2}{2^3}$	$\frac{3^4}{2^6}$		$\frac{3^6}{2^9}$	$\frac{3}{2}$	$\frac{3^3}{2^4}$		$\frac{3^5}{2^7}$	

Up to now there are no problems of homonymy, but when they do occur we see that :

$$\frac{F^{\#}}{G^b} = \frac{C^{\#}}{D^b} = \frac{G^{\#}}{A^b} = \frac{D^{\#}}{E^b} = \frac{A^{\#}}{B^b} = \frac{E^{\#}}{F^b} = \frac{B^{\#}}{C^b} = \varpi$$

We can observe here the intervention of what the mathematician call the **factor system** attached to the section s_p.

Consider the diagram :

$$N \hookrightarrow G \overset{s}{\underset{\varphi}{\leftrightarrows}} \mathbb{Z}$$

with $\varphi \circ s = id_{\mathbb{Z}}$.
The "system of factors" of s is the application :

$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{\nu} N$$

given by :

$$\nu(m,n) := \frac{s(m)s(n)}{s(m+n)}$$

We can check that it is a $2-$cocycle on $\mathbb{Z}$ with values in N. It is a well-known fact that the homology group $H^2(\mathbb{Z}, N)$ is null (see [N] p. 249). So ν is also a $2-$coboundary (which we already remarked since the extension is trivial).

In the case of equal temperament, $N = \{1\}$ and everything is trivial, so there is **no** expressive intonation in a temperate system.

But in the case of $N =< \varpi >$, the choice of the **musical** section s_p gives a non constant **musical** factor system ν_p since :

$$\frac{C\#}{D^b} = \frac{s_p(25)s_p(-24)}{s_p(1)} = \varpi \neq 1.$$

So what musicians do unconsciously is to express their feelings through a factor system !

6) Pure intonation

The "pure scale" (or Zarlino's scale) can be defined in the same frame :

$$\underset{=<\varpi,\delta>}{N} \hookrightarrow \underset{<2,3,5>}{G} \overset{s}{\underset{\varphi}{\leftrightarrows}} \mathbb{Z}$$

where δ means the Didymus (or the syntonic) comma $\dfrac{3^4}{2^4.5}$. The musical section s is defined in the same way as before and we get a similar system of expressive intonation arising from the factor system of our new s, although there are some minor differences with the Pythagorian expressive intonation. Zarlino's scale seems to be well adapted to the works of certain composers like Mozart (see [He], §6 and [Hh] p. 327) whose music requires great harmonic purity.

7) Conclusion

When an artist is faced with a particular score it seems that a natural question he should ask himself is to know in which temperament the composer was hearing (in his inner ear) the symbols he was putting on paper.

In the case of the composers of the second Viennese school (Schœnberg, Berg, Webern) there is no doubt that they were thinking in the equal temperament.

It may happen that some work of our composer has been recorded with his approval, then the style of this performance can give a clue to our question, as it was the case with "Petite Fleur" and its Pythagorian intonation.

In other cases (a certain quartet by Mozart for example) a study of the score in different temperaments might allow one to make a choice which would give the work a greater coherence (see [He] §6).

But these remarks are just concerned by a single aspect of a much broader problem and the reader is referred to [L] for a deeper study.

References

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