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THE THEORY OF CATASTROPHES: SOME EPISTEMOLOGICAL ASPECTS*

"*. . . They constitute the most harmonious moment of all. They show the existence of a preestablished harmony. When a catastrophe occurs, all tensions, everything cancels out.*"[1]

S. Dali [6]

0. INTRODUCTION

The theory of catastrophes will follow the general pattern. It will receive the most flattering praise, as do all theories at their birth. This will be followed by the most acrimonious criticism. And after the enthusiasm of the salons has waned, and the thundering of the specialists has quietened, the judgement of meticulous minds will fall, without recourse, like a guillotine blade.

It is surprising to observe that the various criticisms which have been raised in more or less virulent opposition to the theory of catastrophes do not strike at the foundations upon which this theory has been built. Would it not be natural, however, in order to convince oneself of the solidity of an edifice or to disclose the weak zones where faults might develop, to begin by examining the architecture of the whole, and by testing the solidity of its foundations? An examination of this type would be of greater interest than one such as that of Levy–Leblond [10] which remains at the level of general philosophy, or that of Hector Sussman [14] which, being blinded by the imperfections of certain applications, fails to grasp the intrinsic value of the theory.

Certainly all these criticisms have their value to the extent that they shed some light on weaknesses in the constructions and suggest, either explicitly or implicitly, means for sealing up the breaches, or perhaps even show in a definitive way, the need promptly to abandon a fortress which is headed for certain disaster. But, in order to be complete and fruitful, a critique should also take account of the positive aspects which a work manifests. In this way the critic may avoid partiality and incline more to equanimity and justice in his judgement.

We now describe the basic plan of this study: a presentation of the theory of catastrophes[2], research into its weak points, and into the contributions of the theory. For the sake of convenience of exposition, and in order to avoid a certain dryness of discourse, it will often happen that these two main parts will be imbricated each with the other.

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1. THE CONCEPT OF STABILITY

The theory of catastrophes – this terminology could have been borrowed from geology – rests on certain considerations of natural philosophy which have, from time to time, been given mathematical expression. The basic concept which underlies this theory is that of *stability*. The importance of that idea cannot escape anyone. In order to exist and to be perceived by the senses, an object must possess certain obvious qualities of stability. Embryological processes, for example, reproduce themselves through billions of examples without undergoing important modification: one of their principal characteristics is stability.

Mathematicians have proposed to formalize the notion of stability in several ways. Is it the case that all of these formalizations are well adapted to the different manners in which stability is manifest in the concrete universe? It is for the moment impossible to answer this question because the insistence on considering the notion of stability has come rather from mathematicians than from engineers. In general, these latter have not been able to propose, based on their experiments, criteria of stability of which the mathematician would be able to take advantage.

Thus, there hovers over the theory of catastrophes a fundamental and very light shadow of doubt which bears, not on the principles on which the theory is based, but on the way in which these principles are to be put to use. It is easy to imagine that several notions of stability can exist which are more or less coarse in quality. For each of these, according to the reality for which one wishes to account, a model can be constructed which follows the canons according to which the theory of catastrophes has been established.

2. THE LOCAL MODEL

The passage from local to global is always fraught with difficulties. The theory of catastrophes resolves some, but hardly all of these. We shall not investigate the matter in complete detail because it would be necessary, as a preliminary, to devote oneself to a thorough and critical investigation of the manner in which mathematicians approach this problem. We shall content ourselves with pointing out at opportune moments the existence of difficulties. Before making a case for it, it would be worthwhile initially to examine the local model.

Objects, processes, being localized in space-time, the local model establishes before everything else the spatio-temporal territory of the observable. One is thus led to consider a small domain $W = D \times T$ of space-time which contains the substantial milieu submitted to observation. Allowing oneself a certain verbal licence, one identifies at times either D or W with an elementary part of the substrate milieu itself. The properties of the substrate are assumed to be constant on W. This condition may impose drastic limitations on the size of W; the latter can be chosen, *a priori* as small as one wishes.

This substrate-space is either actually in evolution or already in a stationary state. But even if the state is at the present moment stationary, it was in evolution at some time in the past. Thus, one may always postulate the existence of a local kinetic, or even of a local dynamic $\mathscr{D}_W$ which rules the state of the local substrate W.

On what does this dynamic act? On the parameters which characterize the state of the element of the substrate. At this moment we come face to face with a difficulty which is inherent in every concrete problem: on the basis of what criteria should these parameters be chosen, how are they to be identified? We can only avoid this question here: the answers which it can be given depend on the various ideas which the scholar has of the specific reality which he is studying, the suggestions which his colleagues from other disciplines can supply, the theoretic model which he has in mind and which, to a certain extent, guides his observation, and finally on the possibilities available to him to quantify his observations.

Suppose one was able to prepare the list of these parameters. They would separate into two groups. The first group contains the parameters x_i over which the observer has no direct control: these are the *external parameters*. On the other hand, the experimenter can fix the value of the parameters u_j of the second group: these are the *control parameters*, also called *internal parameters*. The set of values which the external parameters can take is designated by V_W. V_W is called the fiber-manifold over W. The dynamic $\mathscr{D}_W$ acts on the points of V_W.

3. STUDY OF THE DYNAMIC $\mathscr{D}_W$

Let us examine the principal properties of $\mathscr{D}_W$. In the simplest case, the time proper to the different phenomena which unwind at the interior of W is the same. In this case the proper time is one-dimensional. But it is quite easy to conceive of multidimensional time adapted to phenomena which are inhomogeneous with respect to time; the theory of foliations of manifolds treats the case of these dynamics with multidimensional time. In any case, the time τ is a local time[3]; it may be infinitely more rapid than the physicist's time or than human time, so that it often happens that we only observe the final result of the action of $\mathscr{D}_W$; this at least is the point of view adopted in the theory of catastrophes. In each application, the well-foundedness of this viewpoint should be examined with care.

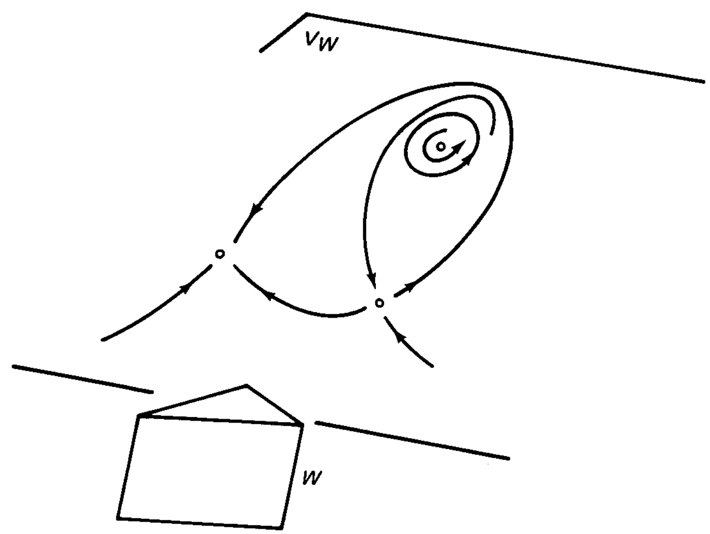

Fig. 1.

The dynamic (as shown in Figure 1) associates to each value of the time τ a point $x(\tau)$ of the parameter space V_W. As τ varies, its image $x(\tau)$ describes a curve called an *orbit* or a *trajectory* of the dynamic $\mathscr{D}_W$. The tangent vectors at each point of all the trajectories defined by this dynamic form a field $\mathscr{X}$ of

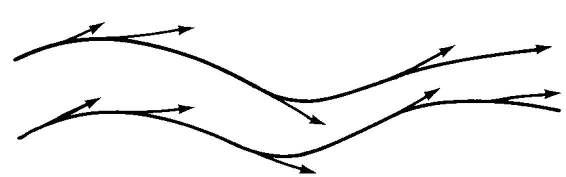

Fig. 2.

vectors X (see Figure 2) – the trajectories are the furrows of a field, the vectors are analogous to stalks of wheat that have been cut down at their base and that lie tangentially to the furrows. It is shown, under certain conditions which are assumed to be satisfied, that the data of these vectors X suffice to reconstitute the system of trajectories which foliate the parameter space V_W.

After a certain period of time, in principle infinite, a trajectory stabilizes itself around an *attractor*: this may be a point, a circle, a torus, etc. If the attractor, which is assumed stable, is a point $\hat{x}$ of V_W, the values of the coordinates of this point represent the possible values of the characteristic parameters of the standard point of W, the substrate space considered. If this attractor is a closed curve, these same values vary periodically in the course of time τ, provided, of course, that the attractor considered is stable. The important thing to keep in mind here is that the stable attractor corresponds to the final result of the action of the dynamic $\overline{\mathscr{D}}_W$, this is the locus of the observables in V_W.

The observed phenomenon presents properties of stability. In order to characterise this state of affairs, mathematicians have introduced the precise notion of structural stability, among others. They posit that a dynamical system $\mathscr{D} = (V, \mathscr{X})$, or even just a system of trajectories, is *structurally stable*, if a slight modification of the parameters which define the components of the vector field does not imply a change in the appearance, in the general pattern of the trajectories. One is not concerned with metric considerations proper to each trajectory; it is only the general equality of the proportions between trajectories which is conserved. This idea of examining problems of stability from the point of view of the *invariance of proportions* deserves, perhaps, to be elaborated.

The classical mathematical formulation of structural stability consists in saying that if $\mathscr{X}^r$ is a field of vectors on V_W which is differentiable to order r (that is to say, of class C^r), it is structurally stable if every vector field $\mathscr{X}'^r$ of the same type as $\mathscr{X}^r$, belonging to some neighborhood of $\mathscr{X}^r$, defines a change of trajectories which is deduced from the first by homeomorphism, that is to say, by a bijective and bicontinuous mapping. Under these circumstances, one says that the dynamical systems $\mathscr{D} = (V, \mathscr{X}^r)$ and $\mathscr{D}' = (V, \mathscr{X}'^r)$, or that the vector fields $\mathscr{X}^r$ and $\mathscr{X}'^r$ are *topologically equivalent* or *topologically conjugate*.

Naturally, it is possible to embellish this definition[4]: to impose limits on the extent of the neighborhood of $\mathscr{X}^r$ from whose interior the $\mathscr{X}'^r$ is chosen, or even to establish limits on the distance which can separate the new trajectory from the old. It is also possible to relax some of the conditions which define classical structural stability. For example, Smale assumes that, instead of applying to the entire system of trajectories, the topological equivalence only bears on the attractors and repellors of the dynamical system: this is the notion of *Ω-structural stability.*[5]

It is clear that, depending on the nature of the concrete problem that one needs to treat, one form of stability will be preferred to another. Thus one has, *a priori*, as many models as acceptable definitions of structural stability, and of possible choices of the dynamic.

4. GRADIENT DYNAMICS AND ELEMENTARY CATASTROPHE THEORY

The simplest dynamical systems have only point attractors: these are the structurally stable ***gradient dynamical systems*** whose vector fields involve only vectors X deriving from potentials (Figure 3).

Let us clarify: let n be the dimension of the space V, and suppose that $X_1, X_2, X_3, \ldots, X_n$ are the components of a vector X of the field tangent to a trajectory in $x = (x_1, x_2, \ldots, x_n)$. By definition, these components have for value:

$$X = \begin{cases} X_1(x) = -\dfrac{\partial f}{\partial x_1}(x_1, x_2, \ldots, x_n) \\ X_2(x) = -\dfrac{\partial f}{\partial x_2}(x_1, x_2, \ldots, x_n) \\ \ldots \\ X_n(x) = -\dfrac{\partial f}{\partial x_n}(x) \end{cases}$$

where the function f is called the 'potential' which assesses at the point $x \in V$ the value $f(x) \in \mathbb{R}$ of the potential to which all points of V are submitted.

Elementary catastrophe theory only concerns itself with gradient dynamics. Thus designated, this may be considered the *simplest* among all possible catastrophe theories. The attention given to this theory can be

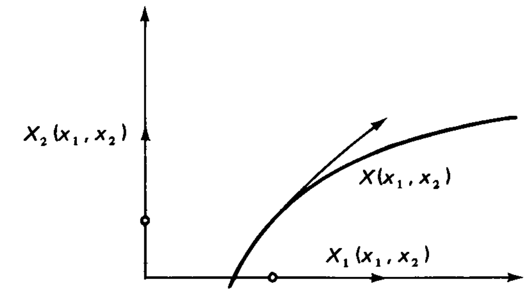

Fig. 3.

justified in the following manner: Experience tells us that Nature, as much as it is able, tends to operate in the simplest ways possible. Since the gradient dynamics possess this quality of simplicity, one may expect that these dynamics are quite common. This is certainly the case in physics: gravitation, electromagnetism and optics, and, at times, thermodynamics.

In the domains where we do not know enough yet to pronounce on the presence of such dynamics it is convenient to begin by postulating their existence. One may adopt the point of view that if this hypothesis should be shown to be false then it is probable that the dynamic realized by Nature will be close to a gradient dynamic. Such a dynamic may then serve as a first approximation to the real dynamic.[6] If in fact the approximation obtained with the gradient dynamic proved to be entirely unsatisfactory then one would opt for a more general dynamic.

We remark along these lines that since the gradient dynamics have the most rudimentary properties, the true 'Taylor' development of a general vector field is not to be obtained by the mechanical introduction of monomials of increasing degree, but by a decomposition of the following type: X = gradient field plus or coupled with a Hamiltonian field plus or coupled with an Anosov field, plus or coupled with a Morse-Smale field, etc. We observe that such a decomposition has already been obtained by J. Roels [13] for vector fields on a symplectic surface.[6]

Recall that, for the structurally stable gradient fields which we shall consider from now on, the only attractors are points (called hyperbolic points). At these points, the potential attains an extremum.

One may ask oneself what conditions f and f' ought to satisfy in order to give birth to two Ω-topologically equivalent dynamical systems. The answer, given by John Guckenheimer [7], is the following: Gradient dynamical systems have their attractors topologically conjugate if f and f' are two realizations on the same universal unfolding F. Before explaining these last terms, we note that this result is crucial in the theory of elementary catastrophes where it is supposed that, passing from local domain W to local domain W' the gradient dynamic does not change its character.

One is now in a better position to recognize the rudimentary nature of elementary catastrophe theory. It is an elementary theory because it supposes the dynamic to be of very simple type – that is, of gradient type; this dynamic is structurally stable over the entire substrate. This permits the *trivial extension of the local model into a global model.* It is clear that a theory as 'naïve' as this cannot pretend to describe the totality of natural phenomena.

In what case is elementary catastrophe theory pertinent? In physics for a start. Physics offers us a small procession of laws to which potentials are associated, and which have an *astounding spatio-temporal stability. This stability appears greater to the extent that the substantial tissue is finer, that the substrate presents firmer characteristics of homogeneity.*

The nature of the ether on which the electromagnetic field acts eludes all imagination. The theory of elementary catastrophes, where potentials reign supreme, is thus perfectly adapted to the study of variations of optical density; the most obvious success of the theory, moreover, has been attained in the study of caustics [1], [7], [8]: experimenters have been able to recover certain nontrivial forms foreseen by the theory. There exist other domains of physics where the texture of the substrate is less tenuous: in a homogeneous phase, for example, whether it be solid, liquid or gaseous; physical law is not entirely of potential type, thus it is natural that, in its application to thermodynamics [17], elementary catastrophe theory would appear as an approximate theory. To the extent that the embryonic milieu divides into homogeneous parts that can be associated to phases, it continues to be possible to use elementary catastrophe theory as we have proposed (cf. [2], [18], [24]; see likewise [4]).

In the cases where the homogeneity of the substrate is softened, the interactions of forces among neighboring inhomogeneous domains wind up frustrating the attempt to employ structurally stable potentials to describe the evolutions of local behavior. In these cases, one can no longer appeal to the models of elementary catastrophe theory.

5. THE NOTION OF UNIVERSAL UNFOLDING

From an historical point of view, it would be possible to date this theory from Weierstass for the preparation theorem and from B. Riemann who had recognised the possibility of classifying mappings and had suggested certain work in complex analytic geometry. The study of the deformations in this geometry (Cartan, Grauert for example, cf. the exposition of Houzel in the Cartan Seminar 1960/61, likewise [12]) led first H. Whitney then R. Thom to examine different aspects of the problem in the real case. Morse theory likewise played a role in the conception of the theory of the universal unfolding. This theory was brought to fruition by J. Mather in the years from 1967 to 1970 in connection with the work of Malgrange, Tougeron, and R. Thom.

Here is how the problem is posed. Let f and f' be the potential functions corresponding to two Ω-topologically conjugate gradient dynamical systems. One seeks to produce a 'versal' function $F(x, u)$ depending on internal parameters $u = (u_1, u_2, \ldots, u_p)$ of such type that by slightly modifying these parameters one can, in the first place, recover either f or f' as one wishes: f and f' are then called *realizations of F. F* should also satisfy the following essential supplementary condition: if the position of the point (x, u) is slightly changed, that of the parameters which characterises the state of the substrate, the value of the potential $F(x, u)$ should vary very little itself. In other words if (x', u') belongs to a neighborhood of (x, u) then $F(x', u')$ should belong to a neighborhood of $F(x, u)$. This discussion can be represented symbolically by Figure 4 h is the mapping that transforms x to

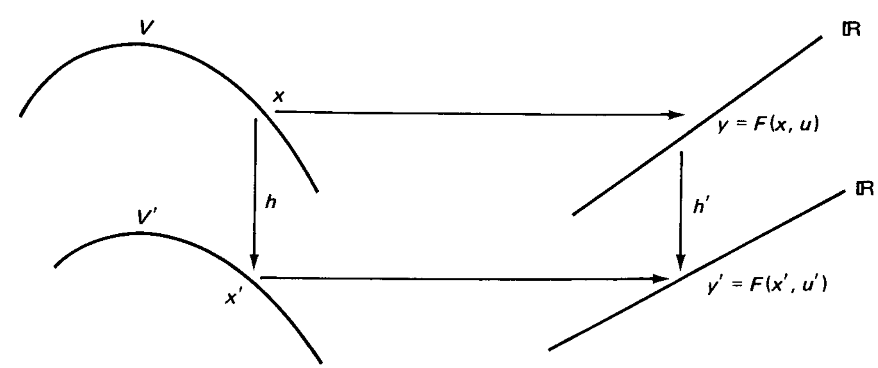

Fig. 4.

x'; this mapping is bijective and, together with its inverse, it is infinitely differentiable; h is called a diffeomorphism; h' is likewise a diffeomorphism; it transforms $F(x, u) = y$ to $F(x', u') = y'$. Among all functions F, there exists precisely one for which the number p of parameters u_i is minimal: this function F, whose expression is $F = f + u_1 g_1 + u_2 g_2 + \cdots + u_p g_p$, is called the ***universal unfolding*** of f. It is demonstrated that the unfolding F can be taken to be in a purely polynomial form. The theory of the universal unfolding can be extended to the case where restrictions are imposed on the differentiability of h and h'.

We should point out the following important property: Morse, Gromoll and Meyer have shown that every function f of $x_1, x_2, \ldots, x_n$, whose derivative vanishes at the origin, – the origin is thus an extremum for f and, as a result, an attractor of the gradient dynamical system defined by the function f – can, in a neighborhood of the origin, be put in the form:

$$f(x_1, x_2, \ldots, x_n) = r(y_1, y_2, \ldots, y_k) + \epsilon_{k+1}x_{k+1}^2 + \epsilon_{k+2}x_{k+2}^2 + \ldots + \epsilon_n x_n^2$$

where $\epsilon_i = \pm 1$ and where y_i come from a change of variables acting on the x_i.

It is seen that the terms in x_i^2 play *no role* in the construction of the universal unfolding of f. Only the function r of the k variables $y_1, \ldots, y_k$ determines the nature of the unfolding, and in the end, the local morphology of the dynamic.

In other words, among the thousands of conceivable or measurable parameters associated with an observation (for example $n = 50{,}000$), the mathematical theory shows that only a small number of them (for example $k = 3$) play an actual role, are relevant, and actually ought to be taken into account. *This reduction theorem justifies the range of possible applications asserted for elementary catastrophe theory.*

Finally we note that the stability property of F may be interpreted in the following way: when (x, u) varies, the total number of extrema of F counted with their multiplicity, should remain *invariant.*[7]

The list of universal unfoldings for which the number of external parameters does not exceed 4 was first established by Thom. The best known of these unfoldings, and the most elementary one, is called the ***Riemann-Hugoniot potential***. Its analytic expression is:

$$F(x, u) = \frac{x^4}{4} + u_2\frac{x^2}{2} + u_3 x.$$

6. THE NOTION OF CATASTROPHE

The major interest of the theory is to *render comprehensible the appearance of discontinuities* in a milieu.

When one passes from a domain W to a domain W', the internal parameters generally change value. There exists in this case a map G: $W \to u$ which corresponds to each elementary domain W a value $u \in R^p$ of the external parameters. We shall see that even continuous variations of u may imply discontinuous changes in the values of the external parameters.

One example will suffice to illustrate this point. Suppose that

$$u = (u_2, u_3) \text{ and } F(x, u) = \frac{x^4}{4} + u_2\frac{x^2}{2} + u_3 x.$$

This potential determines the motion of a one-dimensional parameter x. The latter locates itself at an attractor $\hat{x}$ which is stable or unstable, and at which the derivative $F'_x(\hat{x}, u) = \hat{x}^3 + u_2\hat{x} + u_3 = 0$.

Suppose that the internal parameter u_2 has a fixed negative value. Draw the graph of the function $u_3 = -(x^3 + u_2 x)$ – this graph (Figure 5) is called *the fold*. For a fixed value u_3, the equation $F'_x(x, u) = 0$ can have one, two, or three roots $\hat{x}_1, \hat{x}_2, \hat{x}_3$. These values of x correspond to the extrema of the potential $F(x, u)$ (Figure 6).

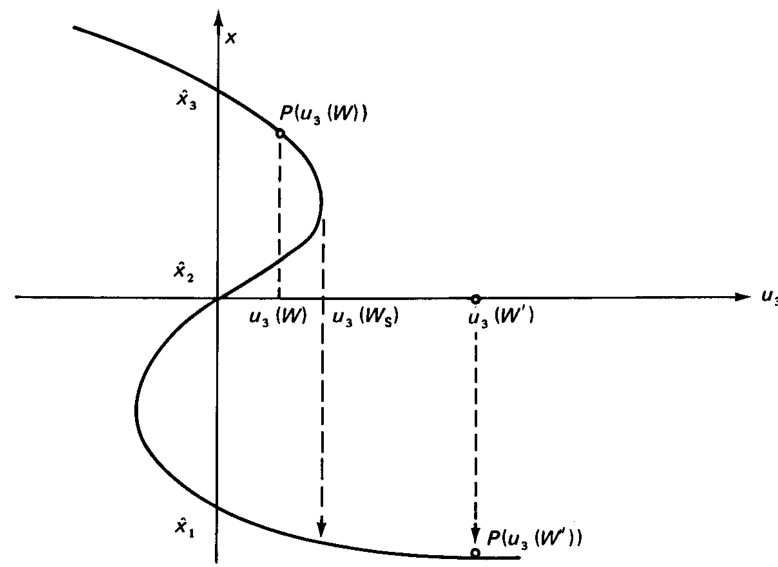

Fig. 5.

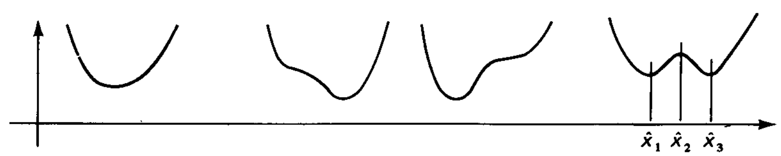

Fig. 6.

Naturally, an extremum such as $\hat{x}_2$, which gives rise to a local maximum of the potential, corresponds to an unstable attractor: it has no assured long-term physical reality in principle, and its presence can hardly even be observed. On the other hand, the values x_1 and x_3 associated with minima of the potential correspond to stable and observable attractors.

Let $P(u_3(W))$ be the point with coordinates $(u_3(W), \hat{x})$ on the fold. Let us suppose that it describes the local state of the substrate W and that displacement to the interior of the substrate from W to W' corresponds to a continuous variation of u_3 from left to right. Notice that, when u_3 crosses the value $u_3(W_s)$, the value of $\hat{x}$ on which the derivative of the potential vanishes passes brusquely from the upper part of the fold to the lower: there is a '*catastrophe*' at $u_3(W_s)$.

Naturally, this jump in the value of $\hat{x}$ in the vicinity of W_s manifests itself in the form of important modifications in the appearance of the substrate. The locus of domains W_s where these local changes of appearance are produced will be called the *domain of accidents A* of the substrate (accident comes from *accedere*, to happen). The image under G of the domain of accidents of the substrate is called the *catastrophe set K* of the formal model which describes the state of the substrate.

The discussion may lead here to the choice of the attractor for the substrate when, *a priori*, several choices are possible. Thom has considered this question by reference to the data of physics. The substrate can 'decide', perhaps with some delay, some 'hesitation', to locate itself at the lowest potential. At this time, there is no formal criterion available for fixing the final choice because, above all, the subtle physical reasons which control that choice are unknown to us. The theory is thus, at the present time, quite flexible: this will give rise to success as well as error in the application of the theory. Let us note once again that the choice of the lowest potential can be made all the more easily as the substrate is more homogeneous.

Another point of controversy concerns the mapping G which is sometimes called the *evolution mapping or the wave of growth*. We know nothing about the types of mappings which Nature uses. It is possible to reason, using an argument from simplicity: G should be at least continuous, and, in a great number of cases, very simple, even close to the identity: the catastrophe set K, calculated by means of a formal model, should give a good picture of the domain of observable accidents A. Hence the idea of saying that if, in Nature, one observes a morphology which resembles that of a catastrophe set, the state of the substrate is ruled by a gradient dynamic whose potential has for expression precisely that which determines the observed catastrophe set.

G has initially been defined as going from the substantial universe to the ideal universe of mathematical entities. Here we recover the first step of the reconstruction by man of the world of ideas. But once this world of ideas is unveiled, it will be necessary to reflect it back towards the world of substantial objects. Mappings H should then be considered that are defined on internal parameters in space-time which associate to each point of a multidimensional catastrophe set a point of $\mathbb{R}^4$ where the morphological accident crystalizes. These mappings H are not arbitrary, but the criteria of their classification have not been uncovered.

We shall end this paragraph by recalling that the universal polynomial unfoldings which are encountered for example in elementary catastrophe theory are standard models. Nature may make use of functions which are topologically equivalent to these polynomials and whose analytic expressions are not very simple. The examination of the pattern of curves can give useful clues: but they can obviously be misleading. It is easy to confuse the graph of $y = x^4$ with that of $y = x^6$; an *S*-shaped curve given *a priori* does not necessarily have a cubic polynomial as model but perhaps an odd-degree polynomial of degree higher than 3. Elaborate mathematical and experimental studies – obtained by variation of different parameters – may, in numerous cases, resolve some of the uncertainty.

7. THE THEORY OF CATASTROPHES, AN UNFINISHED THEORY

Recall the principal points concerning which the theory suffers a certain obscurity. The theory of catastrophes is based on certain definitions of stability: how valuable are they, practically speaking? The theory of catastrophes refers to certain spaces of parameters: on the basis of what criteria should these be defined? The theory of catastrophes makes use of local dynamics: what are the nature and the properties of the local times considered? The theory of elementary catastrophes rests on the use of gradient dynamics: how far can these dynamics remain useful? The gradient dynamics are assumed structurally stable on the entire substrate: to what extent is this assumption admissible? In what cases will the use of non-elementary catastrophe theory be relevant? How should the choice be made from the immense number of possible morphological models? The theory of catastrophes considers mappings between model spaces and substrate spaces: what can be said concerning these mappings?

All of these open questions show to what extent the theory of catastrophes has not yet been completed.

The elementary theory can, we imagine, be extended. More general models can admit: (1) a dynamic of gradient type over the entire substrate; but at one point or another of the substrate this dynamic can undergo evolutions (bifurcations, complexifications, passage from external state variables (resp. internal) to internal state variables (resp. external)). (2) a structurally stable dynamic over the entire substrate but not necessarily of gradient type. (3) a dynamic which is not structurally stable over the entire substrate and of arbitrary type.

Since only the study of dynamics in two dimensions is well advanced, it is readily admitted that the work that remains to be done in order to construct more general catastrophe models is extensive and difficult.

Moreover, the principal results on the properties of arbitrary dynamics leave the theoretician perplexed: the structurally stable dynamical systems are in the end rarer than was originally thought; there are Cantor sets – consequently of infinite cardinality – constituted of stable or unstable attractors, and it is possible to pass in a 'catastrophic' manner from one parameter value x_i to another, and this jump in value can be accompanied by a drastic modification in the nature of the attractor.

It is not yet known, in general, how to identify these mathematical phenomena with physical observables. One might even ask oneself if certain of these statements might not be 'structurally unstable' in the world of ideas, to the extent that they do not correspond to anything tangible in the substantial world. It is necessary to pose this question. The reply can only be made with extreme caution. Because it happens that certain mathematical results only find application centuries after having been established. Because the greatest progress in physics has only been made through the consideration of ideal situations of which we are quite certain that they have no physical reality (frictionless pendulum, etc.). The extension of non-elementary catastrophe theory thus encounters non-trivial difficulties, as much of a mathematical nature as of a physical nature.

Elementary catastrophe theory itself has not finished posing its problems. It is not yet known how to establish the dynamic coupling between internal and external parameters. There is no reason for imposing in each case a trivial passage from the local model to the global model. The number of external parameters need not be limited to 4, the internal parameters to 2. Under these conditions, the catastrophe sets become very complicated multidimensional objects. It is not clearly seen what rules the mappings H obey which permit the passage from these multidimensional model morphologies to the domains of accidents of the substrate spaces.

The theory of catastrophes, even the elementary catastrophes, presents the further inconvenience of lending itself to quantitative study only with difficulty. In many cases, the internal and external parameters, even if it is assumed that they are well known, are not accessible to measurement. Being given the fact that in principle one only observes a single attractor for the local dynamic, whereas there may be many such attractors, it is hardly thinkable that it would be possible to reconstitute these dynamics in the fashion of the paleontologist who tries to reconstruct the entire skeleton of a newly discovered animal starting with a single fragment of bone.

Are we, in the last analysis, in the presence of a true check?

8. SUCCESS OF THE THEORY OF CATASTROPHES

The theory of catastrophes has had certain undeniable successes. It is now a part of a tiny cohort of scientific theories which will yield a better understanding of the nature and the state of things.

This success owes to the wisdom and profound intuition of its founder. Thom saw how to reunite ideas which were already known to create an original synthesis, he knew the appropriate generalization. The Hamiltonian dynamic, for example, uses the notion of fiber space; in this physical theory, the fiber over an elementary domain is equipped with a dynamic characterized by a Hamiltonian vector field; its 'stability' property manifests itself in the form of the invariance of volumes in the course of temporal displacement.

Thom, has, in a certain fashion, proposed some variant and generalized forms of the Hamiltonian theory. The variant results immediately from the works of Morse on critical points and in electrostatics. In the place of a Hamiltonian field, Thom constructs a gradient field acting in the fiber. The property of the invariance of the volume form is replaced by the property of the stability of a potential function. The idea of the theory of elementary catastrophes emerges almost naturally, almost irresistably.

The generalization consists in replacing the Hamiltonian or the gradient dynamic with an arbitrary dynamic, the invariance of the volume form or the stability of the potential by the structural stability of the dynamic. This generalization is a little quick: the Hamiltonian dynamical systems are in general unstable – which does not take away from their interest – in other words, structural stability does not imply the invariance of a volume form, just as the invariance of the volume form does not imply structural stability. Now in numerous concrete cases (liquids, cytoplasms, . . .) volumes remain practically invariant over long periods of time. The properties of dynamics should reflect these constraints determined by observation.

The actual dynamics that should be considered do not have a simple formulation. In fact, they must take account of the presence of many phenomena of wear which lead in general to the elimination of unstable attractors, especially in the case where these form dense sets. Further, these dynamics must take account of the influence of the past as well as that of the future. Modern physicists are coming to reject Hamiltonian systems where the past is unimportant (cf. F. Fer, *L'Irreversibilité*, Gauthier-Villars, Paris, 1977). As far as the role played by the future is concerned, each of us knows the importance of it thanks to the faculties of anticipation and foresight in which we desire to excel. Taking multidimensional time into consideration permits us to account for the concomitant and reciprocal influences of the past and the future on the present in evolution.

Since, incidentally, over the course of decades, centuries, and millennia, we witness evolutions which are marked by growths, by disappearances, by the emergence of new properties, and by restructurations, it is important to know how to retrace the mathematical labyrinth of these evolutions, the progress

and the process at the heart of the disordered litany of bifurcations of multidimensional dynamics.

It can be said that the theory of catastrophes calls up and closely examines a number of quite important mathematical and epistomological problems: it is in this regard a particularly active stimulus for research and reflection. This is not the lowliest merit of the theory of catastrophes.

There is no discipline that will not feel the effect of this theory: it would have known tremendous success and popularity. For it is not possible to trick the intuition of a wide public: The latter has formed the opinion, as, for example, has Salvador Dali [6], that the theory of catastrophes offers something refreshingly new.

Its contribution is made at three levels: that of concepts, that of observation, and that of comprehension and representation.

On the conceptual plane, the theory of catastrophes will not only spread a certain number of useful and relevant mathematical notions and approaches, but above all, it will permit a general understanding of the major interest which the concept of stability presents. It is worthwhile to remark here that the theory of catastrophes discusses the dichotomy which appears to us to be of central importance: permanent–transitory, or stable-unstable for which the form continuous-discontinuous is often only an avatar. Thus, the theory of catastrophes enriches our way of thinking of a dynamic concept.

Clearly this new way of thinking leads to a transformation of our habits of observation. In the first place, the theory of catastrophes, insofar as it is a morphological theory has given new impetus to the observation of forms. Questions of why and how are raised anew with respect to them. In the second place, observation is now taking a very close interest in evolutions, it devotes itself to distinguishing between transitory, unstable phenomena and permanent, stable phenomena. The presence of the model of the theory of catastrophes leads experimentalists to research new explanatory factors, to try to give evidence of rough evolutions. It is known now, for example, that the consideration of a single parameter is of no value in general for the understanding of a phenomenon. On the other hand, if one has two control parameters u_2 and u_3 there often exists, when they are varied simultaneously in the (u_2, u_3) plane, a curve in the neighborhood of which qualitative variations of form, of speed of transformation, suddenly appear. This theoretical knowledge affects the activities of experimentalists, whether they are biologists, specialists in nuclear physics [9], or in strength of materials: it allows them to undertake experiments which illustrate the grand principles on which catastrophe theory is founded [21], [22]. By restoring their vigor and modernising the processes of observation, the theory of catastrophes thus lays the foundation for new discoveries, new scientific insights.

On the level of comprehension, the theory of catastrophes helps us see how and why discontinuities or faults can appear at the heart of a single milieu. This contribution to the understanding of phenomena is far from being negligable, even if, *a posteriori*, the solution given by this theory to the problem of discontinuities appears elementary. To speak the truth, the possibility of sharp variations in the value of a parameter has been known for some time. This mathematical phenomenon had already been observed by Lienard for example almost 50 years ago [11]. But Thom, by boldly creating a physical theory, knew how to demonstrate the entire extent to which advantage could be taken of simple dynamic considerations.

One of the astounding interests of the theory of catastrophes is, moreover, the structural unity which it offers to very diverse phenomena belonging to the farthest realms of Nature. The classical structuralist theory is a static theory: one there analyses the internal organization of objects in an a-temporal optic. The theory of catastrophes encourages a structural research in an evolutive perspective. The interest is in discovering the dynamic internal organization of objects. At this moment, hidden analogies of situation and of behavior are coming to light. The too exclusively rational mind will not succeed in perceiving nor in grasping secret dynamic similarities, and will be unable to enjoy this intense satisfaction which the presence of a coherent model which subsumes all phenomena can supply. This model symbolises the profound unity which we perceive in Nature, it is the expression of it, and gives birth to a primary feeling of comprehension of the set of natural phenomena. By thus disclosing the universal which is hidden behind the particular case, the theory of catastrophes reveals itself as an eminently scientific theory.

One may ask oneself whence comes this assurance which the adepts of the theory of catastrophes plainly evidence. We attribute it, for our part, to the geometric intuition which establishes an equivalence between the structure of the substantial world and that of the world of ideas, and which has permitted, thanks to the fundamental work of Whitney and of Thom, a glimpse of a

piece of the architecture of the world of mathematical ideas: we are beginning to have a clear understanding of the stratified manner in which function spaces are constructed. The potential functions of Thom are placed at the base of this edifice. By virtue of the ontological argument of simplicity, we are led to believe that Nature uses potential functions more frequently than others. The hesitation to use these functions, prejudices that we hold concerning them will wear away. This does not amount to saying that we shall allow ourselves to be dazzled by these few elementary functions to the point of believing that everything in Nature reduces itself to their simple consideration.

This ontological argument, the reasons which we have presented in examining the question of the relevance of elementary catastrophe theory explains in our view the practical success of the theory: to distinguish certain forms which one would not otherwise be able to recognize (caustics, radiology), to discern or interpret the rough variations of internal structure (strength of materials, changes of 'phase', geomorphological accidents), to reduce the arbitrariness of description by making use of standard models.

On this point, we note that certain people express disappointment at the fact that the theory of elementary catastrophes is in the end only a phenomenological and descriptive theory. This reaction seems to us to be slightly superficial. This is not the place to undertake an extensive study on the nature of explanation. On the one hand, if one investigates it closely, one sees that the latter is often confused with description. On the other hand, a good explanation is not possible without some descriptive apparatus adapted to the properties of the object which one wishes to understand. The theory of catastrophes, for which observed organization is only the result of a dynamic process, offers a tool for the description of certain forms, it allows their classification and provides some understanding for the reasons for their accidents. When it is a matter of simple forms, like those tied to the Riemann-Hugoniot potential, the theory hardly provides anything new because our minds, as poorly educated in the matter as they may be, are capable of grasping quickly the different manners in which a simple form can evolve. But as soon as one is in the presence of more complex forms, theory and mathematical calculation are necessary to describe the multiplicity of possible configurations: in this case, the predictive character of the theory of catastrophes affirms itself.

It might be remarked that in a good number of applications of the theory the complete model which we have presented is largely edulcorated. Quite often, in a doubtlessly incorrect fashion, attention is no longer given to the presence of the substrate space. Local time is ordinary time. The theory is only utilized in the aspect of universal unfolding. A given phenomenon, for example, the bending-effect [9] is associated to a curve. The stability of the phenomenon is characterized by the fact that this curve is the graph of a 'stable' or 'invariant' function in the sense of the theory of the universal unfolding (paragraph 5). The variation of this curve as a function of parameters which play the role of control parameters is studied. At the same time, the attempt is made to find the values for which catastrophic events appear. Such uses of the theory of catastrophes are frequent. They are encountered everywhere in the experimental disciplines.

In non-experimental disciplines, the use of the theory is slightly different. This theory permits the codification of certain dynamic schemes which press behind all activities, and behind all natural constructions. In the majority of cases, and at the present time, these dynamic schemes are hardly accessible to experimentation. Intuition perceives their existence and some of their principal traits. Lacking precise data and as a first approximation, they are identified with the dynamic schemes inspired by elementary catastrophe theory. It is highly probable that a good number of these identifications are abusive and misplaced. But for the first time in our history, thanks to the mathematical metaphor, we have the possibility of tapping the millennial vein of conceptual organizations, of schemes of mental action. A new symbolic is set in place which uses a single language to describe narrative, linguistic and figurative structures.

The value of this symbolic is subject to caution, and there will certainly be no lack of good minds, of a positive inclination, who will strip it of all value. That is, they will turn their noses up at the intuitions of those who make use of this symbolic. This would be to forget that the development of knowledge passes through preparatory phases where the imagination unfolds itself in all directions, before ending by enclosing the subtlest constructions of life in finer and finer and stronger and stronger coats of mail. History will thank the theory of catastrophes for having permitted the forces of the creative irrational to escape from the enclosures in which classical theories held them bound.

9. CONCLUSION

The balance between the contributions and the weaknesses of the theory of catastrophes tips heavily in favor of the former. Science does not operate with simple measuring devices alone. It makes at least as much use of its conceptual tools. The theory of catastrophes offers at this level a new device. It cannot explain everything of itself, but, by orienting observation in a new and dynamic manner, it gives a glimpse of enriched perspectives of development.

What good is quantitative prediction in those domains where qualitative prediction is still beyond our reach, not that this does them any dishonor. Astronomy is one of the most beautiful flowers of Science; it is also one of the most ancient. We are prone to forget how many millennia separate Ptolomy from the first men who had looked questioningly at the sky. Between Ptolemy's system and Einstein's theory there stretch seventeen centuries. And despite everything, in spite of the successes of empirical astronomy and mathematics, the simple three body problem has not today been given a definitive solution. The impatience which certain people show in desiring that a single idea might resolve at one gulp, in a few hours, the totality of problems which Science poses does not weigh in favor of the realism and the good scientific sense of these scholars. The slightly negative attitude which Thom has with respect to the predictive value of the theory of catastrophes on the quantitative plane, whatever may be the reasons which accompany this position, are not unjustified when one considers the present status of experimental information and of the theory of catastrophes itself.

In its present youthful state, this theory is totipotent, like a germinal cell. It contains the best and the worst. But, as a child of the ages, it will take an important place in experimentation, its domain of validity will be better extended, its field of application better defined. The techniques, the possibilities, the criteria, and the objectives of observation will have changed. What will be the stature in seventeen centuries of the theory of catastrophes, how will it have evolved?

Every effort of human thought is oriented towards simulation of the environment, for as much attention as one gives it, each day, the limits are extended. This human effort appears in agreement with the pattern of the general progress of animal development. To become aware of this current that carries us should someday offer an explanation of the reason for being. Then we shall be able to ask whether it is better to struggle against it or to follow it; whether, in the end, we are the pawns in a chess game, or gods in the process of becoming. Is the theory of catastrophes no more than a toy in the hands of gods, or does it rather appear as the expression of a divine form which allows us to play with it? Like the toys given to children to help them better know the world. . . .

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NOTES

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[1] This point of view is sometimes true.

[2] It is also developed in [15], [16], [3]. Reference [4] contains in an appendix a presentation which is accessible to non-mathematicians.

[3] With the aim of clarifying a bit, in my mind, the conceptual scheme at the heart of which the theory is constructed, the idea came to me in 1970 to introduce the notion of local time. Although this notion has not yet become, even at the present time, very explicit, it appeared to me to be useful to keep it while writing the Seminar given by Thom on the *Modeles Mathématiques de la Morphogénèse* (IHES, 1970–71). The text of this seminar has been published in the *Lezione Fermiane* and in [16]. Naturally, it will be desirable to know some precise examples of relations between different local times and ordinary physical time.

[4] It may be remarked, incidentally, that the definition of structural stability, by taking into consideration arbitrary small variations of the parameters, brings back the study of certain types of stochastic dynamical systems to a geometric and deterministic study of very general dynamical systems.

[5] In fact, Ω designates the set of non-wandering points of the dynamical system.

[6] *Added in proof:* I conjecture that Roels' lemma can be extended to any dimension for physical reasons: any vector field is the sum of a gradient vector field and a conservative vector field. Suppose that the vector field rules the evolution of a specific phenomenon. If the Hamiltonian component is of reduced weight compared to the gradient component, the use of elementary catastrophe theory yields a good approximation to the study of the phenomenon. In fact, this is often the case: since the gradient systems are fairly stable structurally, while the Hamiltonian systems are fairly unstable structurally, then if the studied phenomenon shows some stability, its gradient component must impose over the Hamiltonian component.

[7] This result, conjectured by the author in 1970, has been published independently by Damon and Calligo in 1974, see [5].

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