THE POINCARÉ SURPRISES

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Abstract

A pedagogical project, called *the ARPAM project*, has been founded in an attempt to foster a rapprochement between the general public and the mathematical sciences. It is based on the presentation of mathematical concepts and facts through some buildings called *follies*. This paper describes a folly devoted to the illustration in the physical world of two important concepts, those of singularity and bifurcation, linked through the fundamental concept of stability. The use of properties of some fluid flows on certain surfaces is the main tool which is used to exhibit these concepts and mathematical facts. The artistic attraction of this folly comes from the multiple effects of light through the various uses of the fluid elements.

1. Introduction

The ARPAM project, that was briefly described in the Maubeuge Colloquium on Mathematics and Art ([1] and www.arpam.free.fr), primarily consists of small buildings called *follies*, which are approximately the size of a private house, and whose shapes are tightly bound with mathematics. They illustrate mathematical concepts and facts from a historical viewpoint in the following sense : each folly being linked with the start of a new mathematical theory, the follies are laid out in an approximate historical order. The way in which a tourist should visit the park is thus bound with the history of mathematics. The main scope of the project is to help foster a rapprochement of the general public with mathematics. However, the project might also be of some interest in architecture when it offers new shapes and new subjects of decoration.

Most mathematicians are familiar with the shapes and the ideas behind the decoration of the follies. But that is not at all the case for the general public who may find them quite unusual. The oddity of these shapes and decorations, and often their beauty, are thus expected to be a source of attraction for the general public. Through these incentives and experiences, the public will be able to get in touch with the essential natural data that justify the interest of mathematics, and lie at its root.

The concept of stability could be the most important universal concept in natural science. An intelligence as keen as Plato's did not ignore it. It appears in his natural philosophy when Diotime says in the Banquet that "the mortal nature always tries, as much as it can, to reach perpetuity and immortality", an aim that has been generalized by Spinoza who said that "any thing … always remains steadfast in one's state". This meta-physical notion takes many faces and indeed underlies all mathematical activities.

The follies I have presented until now are mainly linked with the static aspects of mathematics. Static states can be understood as the final stable stages of evolutions. These evolutions can be very fast or very slow according to the temporal scale of observation. The fact is that they underlie the birth, morphology, and activities of all objects in Nature. Any attempt at popularizing mathematics cannot forget to exhibit the analysis of the incarnation of movement into the physical world made by mathematicians. This analysis has given rise to the extremely important cinematic and dynamical theories that are present within the mathematical world.

The history of these theories begins with Galileo and Newton. From that time until Poincaré, efficient mathematical tools were developed to master the analysis of motion : series, differential and partial differential calculus. While these tools do have a geometrical substratum, it is usually hidden in contemporary teaching. Thus, these tools seem to be based only on numerical laws and algorithms. Until Poincaré, during the second half of the eighteenth century, there was no general geometrical theory directed to the study of motion which was able to show and to classify the various behaviours of the trajectories characterizing the different families of motions. Poincaré introduced such a theory and began to study it, at first when the motion was on 2-dimensional surfaces.

The purpose of the folly titled the *Poincaré surprises* that has not been described before, is to visualize some of the main concepts mainly through the exhibition of suitable families of physical trajectories.

The realization of this folly is linked with many technical problems that need only be described if the construction of this folly is effectively considered, and that is not the case for the present. Thus I shall only give here a short general description of this project.

2. Conceptual and Physical Background of the Folly

The folly is built around the concepts of movement and stability, and some of their main by-products. Two of them are particularly significant :

a) S*ingularity* : a singularity can be viewed as an object which has the highest stability. With respect to time, this high stability insures some invariance, some fixity. It can be characterized by high values of some of the parameters by which it is characterized. In such a way, a singularity of a given type usually has the important property to organize the space around it in a specific manner. For example, think of a president of an association or a state : with respect to the power of a person in an assembly, he occupies a singular position, the highest, and models his administration. Singularities are rare : not many people in a country have been elected as president of the state. A cathedral is a very singular building among many others in a city. More generally, very beautiful objects, paintings, jewels, sculptures are rare, precious, and these qualities are attached to their singularity.
Mathematical examples of basic singularities arise from hydrodynamic and geographic analogies : rivers are analogous to trajectories, and singular trajectories which are reduced to points, are called sources, sinks, and saddle points. The meaning of the terms source and sink is clear : from a source, rivers or trajectories goes out, while they end their lives at a sink. A saddle point has both the properties of a source and the properties of a sink : the river or trajectory may be first directed to that point which acts as a sink, but then, approaching the saddle-point which acts now as a source, it is repelled towards another direction. These singularities will be illustrated by the set of basins and fountains surrounding the main body of the folly (see the paragraph **3.3**).

b) *Bifurcation* : bifurcations occur immediately after a more or less partial transitory destabilisation and destruction. An object which is loosing its previous properties is of course close to disappearing : its stability can be very low. The object is then in a very transient unstable state which is also singular, but for opposite reasons to the ones stated in the previous paragraph. When the object does not vanish and then gives up this dramatic situation, it recovers a new stable state which can evolves. This new state is defined by some quite new values of the parameters which characterize the state of the object. The term bifurcation refers to these very fast and important changes of these characteristic properties of the object. Classically, in physics, the fast changes of phases, say from the liquid state to vapour state, is the paradigm of the bifurcation process. Metamorphosis can also be understood as a bifurcation process. The evolution of societies is going with revolutions and some wars which look very short respect to the historical time : these societies may be considered as singular during these periods, after which they keep on their evolution in new ways.

More generally, a bifurcation means a change in the characteristics of a singularity, which may unfold in several determined ways, when it is of course under a complete control.

Let us introduce here first some classical examples of mathematical bifurcations that will be exhibited in the folly.

Invariant subspaces with respect to a parameter, like time or any other physical attribute, are of course the most stable parts of space with respect to this parameter. Punctual singularities are singularities in the family of general invariant subspaces.

These punctual singularities belong to an important sub-family of frequently encountered invariant subspaces called *tori*. A point is also called a torus $\mathbf{T}_0$ of zero dimension. A circle, that can be seen as an invariant trajectory, is a one-dimensional torus $\mathbf{T}_1$: such a circle represents a periodic movement ; the length of the circle can represent the length of time at the end of which the object gets again its initial position or state. If the radius of the circle $\mathbf{T}_1$ vanishes with the time, the final result of this shrinking is the point $\mathbf{T}_0$: in that case, the periodicity of the movement is null. Conversely, through a bifurcation process, the singularity $\mathbf{T}_0$ may *unfold* into a circle $\mathbf{T}_1$.

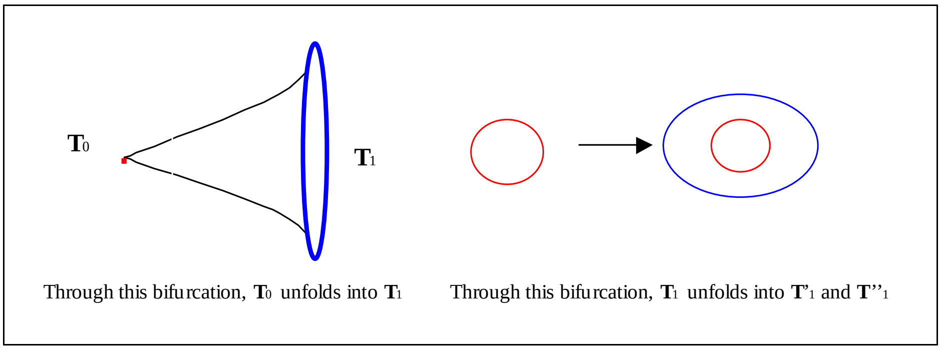


Figure 1 : *Two theoretical bifurcation processes*

When, in its turn, each point of this circle unfolds into a circle, then we get the ordinary torus $\mathbf{T}_2$. Inductively, the unfolding of the points of $\mathbf{T}_{n-1}$ gives rise to the n-torus $\mathbf{T}_n$ which is a representation of complex periodic movements. But from the circle, an other bifurcation process may happen, $\mathbf{T}_1$ giving birth to two circles $\mathbf{T'}_1$ and $\mathbf{T''}_1$.

The bifurcation $\mathbf{T}_0$ towards $\mathbf{T}_1$ means the transformation from a steady state to a periodic behaviour. It frequently bears the name of *Hopf' bifurcation*. It plays an important role in the birth of the "Bénard cells and rolls". Take a liquid between two horizontal plates, and heat the lower one. When the temperature reaches a critical value, one observes that the space between the two plates divides into cells, and inside each cell, the liquid moves along a closed line as the one of a circle or an ellipse.

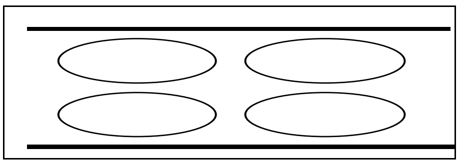


Figure 2 : *Bénard cells and rolls*

Bifurcations induce changes in the behaviour of trajectories. In the extreme cases, they may be changed for instance into Cantor sets of singular points, which globally can be stable. They may also be very unstable giving rise to a turbulent behaviour.

Some Cantor sets in the plane can be represented by apparent arcs of curves : in fact, if we were able to have an infinite zoom, we should see that these arcs are made of an infinity of distinct points, so that there is always empty space between any two points. The first algorithm to construct a Cantor set, the middle thirds Cantorset, has been given by Cantor himself. This algorithm, all the algorithms created for this purpose as well, has no finite end. Thus, the use of exact algorithms can exhibit but approximations of Cantor sets, as the following one I am indebted to Mike Field[1] : "My idea originally was to do a cone on the middle thirds Cantor set. I played with things a bit and decided that mapping the Cantor set linearly onto the arc [Pi/4,7Pi/4] and forming the cone could give a nice representation". See :

http://nothung.math.uh.edu/~mike/cantor.tif

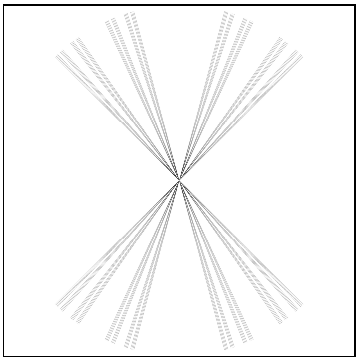


Figure 3 : *Part of the cone on the original Cantor set*

Since several terms of dynamical systems theory come from situations observed on the surface of the earth, the conception of this folly is based on a reproduction of some of the most significant geographical situations in a flexible way, through physical models of the water flows. These situations will allow to illustrate the terminology used by mathematicians, a few facts as well. Simulations of the behaviour of these fluids are being made by Stefan Turek from the University of Dortmund <http://www.mathematik.uni-dortmund.de/htmldata1/featflow/>.

3. Short Description of the Main Body of the Folly

3.1 The central ring. The main building has a rotating axis of symmetry, which is inside an optimal designed column [2]. On the ground, inside the building, in principle, it moves a cylinder which is the inner vertical border of a basin having the shape of a ring. This basin will be named the *Poincaré-Birkhoff ring*.

Its outer vertical border (radius 180 cm) should be turning in the opposite sense of the inner cylinder (radius 60 cm) : then, by friction and capillarity, the water in the basin, close to the borders of the cylinder, is moved in opposite directions. From time to time, a fluorescent jet of water comes out tangentially to each cylinder ; for instance, the one coming out from the inner cylinder is red, while the one coming out from the outer cylinder is green. In fact, here, the basin will be fixed, and the water will be put in movement by the actions of the jets. According to the speeds of the borders, or equivalently to the power of the jets, different behaviours of strings of water will be observed along the circled located in the middle of the ring.

[1] I thank him also for having greatly improved the English of this article - under the right pressure of the referee!

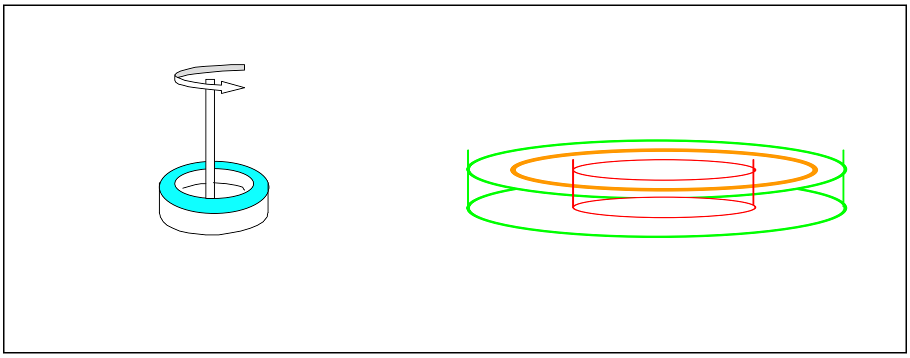


Figure 4 : *The Poincaré-Birkhoff ring*

In particular, we get a physical illustration of a theorem proved by G. Birkhoff that there exists an invariant trajectory in the middle of the basin. Hereafter are theoretical illustrations of two possible observations : the spectator will attend to the birth of invariant flow, here coloured orange, in the middle of the basin.

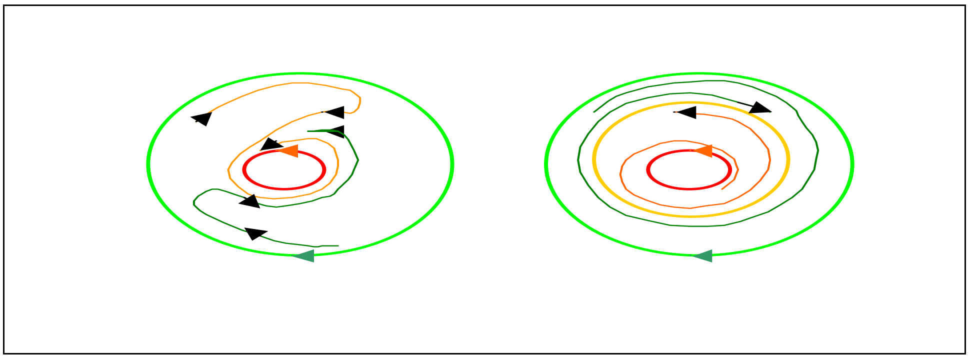


Figure 5 : *Illustration of some theoretically observed flows*

3.2 The roof. It has two parts : a fixed part and a moving part just above the fixed part. The rotating axis makes moving the upper part of the roof which is like a wheel. It is transparent, so that one can observe fluids which are flowing between the two parts. These fluids, which can be coloured, are pumped and are coming up inside the axis which also acts as a pipe. They are sent under pressure between the moving and the fixed roofs.

The water goes off along the border of the roof , except above the entrance. First there will be a gutter to collect water when it is raining, but also the liquids between the two roofs will not reach this part, called the "blind sector", being stopped by a kind of fixed barrier inserted on the fixed roof.

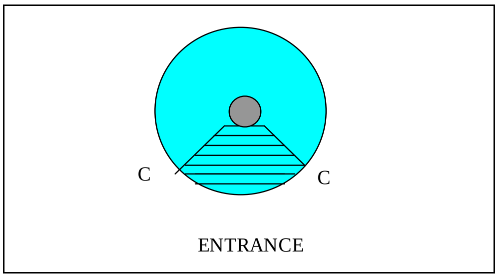


Figure 6 : *No fluid in the blind (hatched) sector*

The radius of the moving roof should be around 450 cm, the angle of the blind sector is 60 degrees which gives a protected part for the entrance about 470 cm long.

Through a system of small balls, the moving roof rolls upon the fixed roof. The speed of rotation can be changed. The distance from the fixed roof to the moving roof is around 10 mm at 30 cm from the mathematical axis of rotation. This distance is around 2 mm along the border of the roof.

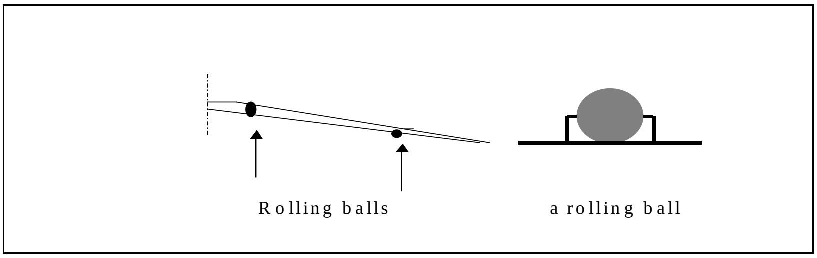


Figure 7 : *Principle of the rolling system*

There are many ways to fix up the rolling system, which will have a great impact on the sight given by the evolution of the trajectories. The horizontal axis of the rolling balls is supposed to be 1cm large ; they have to be shaped according to their radii.

Here is a tableau giving a possible arrangement of the rolling balls :

radius in cm	number of rolling balls	angle (degrees) between two consecutive balls
30	4	90
90	8	45
150	12	30
210	24	15
270	24	15
330	30	12
390	40	9
450	50	7.5

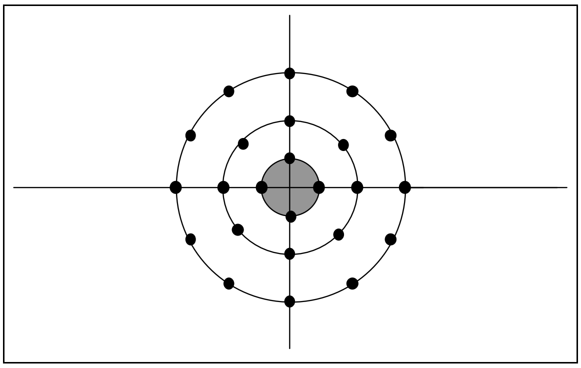


Figure 8 : *Positions of the rolling balls on the first three circles of the moving roof*

Being located above the building, we can thus observe the surprising coloured trajectories on the roofs of the building. All the bifurcations processes described before will appear. They will vary with the pressure of the fluids, the speed of rotation of the upper roof, the exterior temperature and the climatic conditions. Falling down, the water makes a kind of crystalline bright curtain around the building.

3.3 This water falls down in some basins surrounding the main body of the building. One may imagine these basins as the little chapels forming the counterforts of a cathedral.

The sources are located at the highest places of this construction. Coloured water can spring from them. From time to time jets of water like geysers are springing up. Saddle-points mark the limits of the influence of these sources. Sinks collect the water.

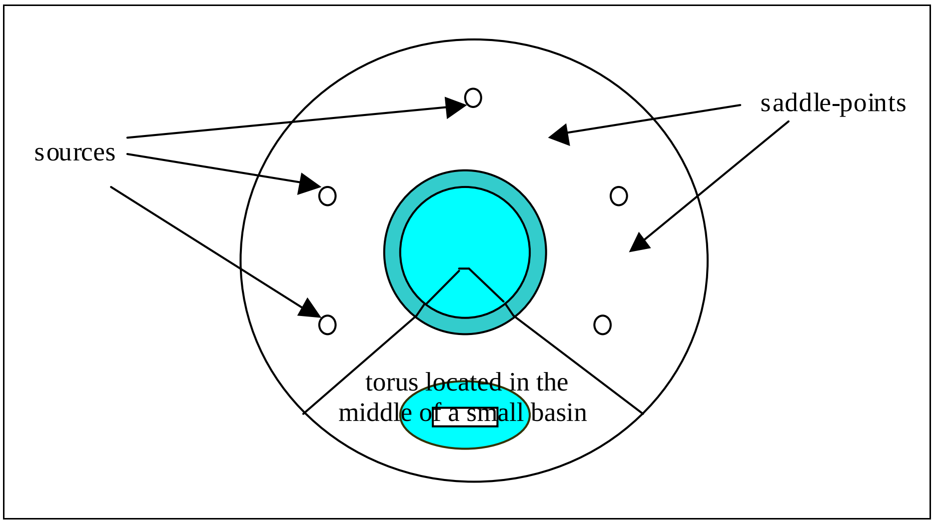


Figure 9 : *General disposition of the complete folly*

3.4 A 2-torus is placed before the entrance, in a small basin. There is a source at its top. We expect that the properties of capillarity will allow the liquid to cover all the surface in order to exhibit the usual singularities of the flow on the torus which are pointed out on the following drawing.

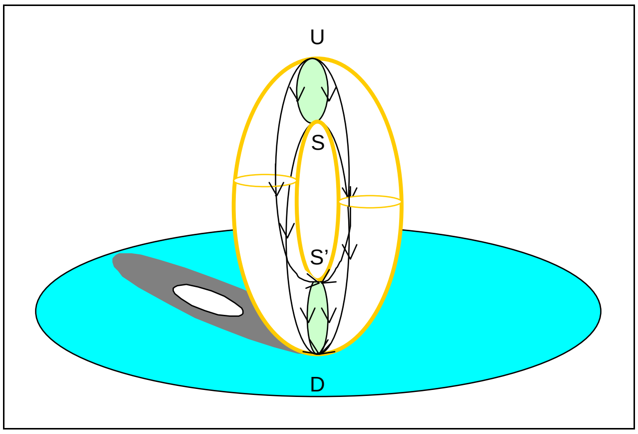


Figure 10 : *The stable flow on the 2-torus (*U *is source,* D *is a sink, while* S *and* S' *are saddle-points)*

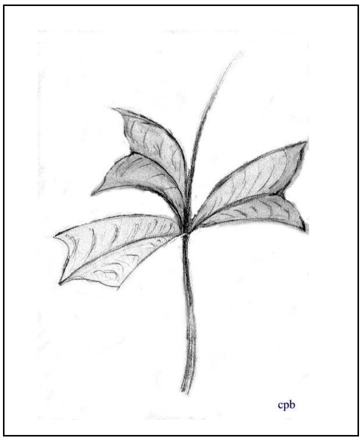
4. Other Indications for the Decoration

4.1 Inside the folly, animations, physical and chemical devices, which have now become rather classical, will show bifurcation processes and chaotic behaviours. Simulations of the movement of waves will also appear on some screens ; this movement is bound with the theory of solitons which do have the stability property (for example [3] and the exhibition :
http://rsp.math.brandeis.edu/3D-XplorMath/TopLevel/gallery.html (pseudo-spherical surfaces)).

4.2 Decoration will show fixed or animated sequences of computed trajectories and phase-portraits. They can give rise to beautiful pictures. For instance, look at some works by Tim Stilson :
http://www-ccrma.stanford.edu/~stilti/images/chaotic_attractors/nav.html

4.3 The outline of the main body of the folly is a part of the hyperboloid of one sheet. This ruled surface is commonly used in architecture to make a water-tower. Its outside will be tiled with imitations of very large quartz crystals : they will reflect and diffract the light coming from projectors located around the main body of the folly. These projectors will also illuminate the bright curtain of fluids falling down from the roof.

4.4 Finally, above the central fixed part of the roof, a sculpture will symbolize the miracle of stability through an unfolding of a singularity. This sculpture will be a kind of flower with three leaves. Each leaf is viewed as a deformation of a half part of the Whitney umbrella.



References

[1] C.-P. Bruter (Ed.), *Mathematics and Art*, Springer-Verlag, Berlin Heidelberg, 2002.
[2] S.J. Cox and L. Overton, *On the optimal design of columns against buckling*, SIAM J. on Math. Anal., vol. 23, pp. 287-325, 1992.
[3] R. Palais, *The symmetry of solitons*, Bulletin of the Am. Math . Society (NS), vol. 34, pp.339-403, 1997.