ON THE NOTION OF ORIENTATION

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I. Introduction

The notion of orientation has been and is used in many contexts, under different names, and there are different manners to understand its history. Some people prefer to remain within a specified discipline ; others like to study how a notion evolves inside a field, then influences some connected domains, and sometimes comes back to the original field with the enrichments given by the connected domains, including of course the change of name. One may also wish to trace back the concept in the works of authors where it is present though used in an informal way, maybe under other denominations. But one may refute this approach because, sometimes, the subjectivity of the reader can be involved.

Some people again prefer the exposition of pure and crude facts, so that there cannot be any refutation. Other seem more interested in following the causes and the processes of evolutions, and in the lessons that one can extract from these analysis.

Each of these points of view seems to be well-founded. But in the present case, to have a separate treatment of each of the points is impossible. This paper must be understood as a first attempt to clarify a fairly complicated situation.

One of the difficulty arises from the fact that we are given a concept, and that this concept has appeared under different names like « order », « sense » or « sens », « direction » and « Richtung », « orientation ». An other difficulty comes from the fact that, given sometimes the obviousness of the situation, the formalization of the concept was felt as unnecessary by the first authors.

I shall concentrate mainly on the mathematical aspects, being interested by the concept in relation with the term « orientation .

Before entering the subject, I would like to say, from the mathematical side, why an historical study of the notion of orientation has some interest. Because the orientation has a key position, first in the study of integration since non-orientability can be an obstruction to integration, second in the study of the properties and the classification of manifolds. Let us also recall the place of the orientation in the Lie derivation, and in Finsler geometry where the metric on the manifold depends not only on the given point on the manifold, but equally on the direction choosen in the tangent space in that point.

II. The topological orientation

The distinction between the topological and the geometrical orientation has been made by Élie Cartan in his 1941's conference to the French Mathematical Society, *La notion d'orientation dans les différentes géométries* [14]. We shall deal here with the topological notion of orientation, nowadays defined in every course in differential geometry.

In his book *La topologie algébrique des origines à Poincaré* [43], J. C. Pont allows us to follow the evolution of early combinatorial topology and the importance of the notion of orientation in this development. In the section V of this paper, I shall give some complementary references showing the implicit presence of the concept in the mind of other mathematicians. For the subject we are concerned with, Pont refers mainly to Euler, Meister, Lhuillier, Gauss, Listing, Möbius, Van Dyck. It should be underline that *none* of these authors uses the term « orientation » or « oriented ».

Three works are particularly significant for our purpose. The ones of Listing and Möbius who, in 1858, discover the first « unilatère » (one-sided) surface, the so-called Möbius band, and the work by van Dyck in 1890 where, for the first time, the determinant is used to evaluate what now we call the orientation :

« A continuous transformation of the directions of the axes of coordinates, along a closed curve, can be analytically represented by a linear transformation depending on a parameter ; when, after a complete turn, some axes have taken back their initial position, a part of the determinant of the transformation has taken its initial value, with the same sign or with an opposite sign, according to the number of axes which have changed of direction is even or odd. » (Quoted by J.C. Pont).

Let us notice that the functional determinant has been introduced by Jacobi in his article *De determinantibus functionalibus*, published in 1841 in the Journal de Crelle [30] :

« Propositis variabilium $x, x_1, \ldots, x_n$ functionibus totidem $f, f_1, \ldots, f_n$, formentur omnium differentiala partiale omnium variabilium respectu sumta, unde prodeunt $(n + 1)^2$ quantitates

$$\frac{\partial f_i}{\partial x_k}$$

Determinans ad harun systema pertinens

$$\sum \pm \frac{\partial f}{\partial x} . \frac{\partial f_1}{\partial x_1} \ldots \frac{\partial f_n}{\partial x_n}$$

voco *Determinans functionale* vel, magis diserte, Determinans ad functiones $f, f_1, \ldots, f_n$ variabilium pertinens sive functionum $f, f_1, \ldots, f_n$ Determinans $x, x_1, \ldots, x_n$ respectu formatum. »

We can consider that it is Poincaré who, in his famous article *Analysis Situs* in 1895 [42], defines in the modern way both the notion of manifold (variété), and the one of orientable manifold. The fact that he remains in the frame of the analytical varieties and that some of his terminology is out of fashion has no real importance. The extension of Poincaré's definition of what is now called topological orientation to differential manifolds and to fiber spaces may be considered as trivial. All these definitions have the same conceptual meaning.

The term of variety was introduced by Gauss in a work titled *Theoria Residuarum biquadraticorum, Commentatio secunda*, presented in 1831 and published in Göttingen in 1832 [24] :

« Si, a conceptibus, quos offerunt varietates duarum dimensionum, (quales in maxime puritate conspiciuntur in intuitionibus spatii) profecti, quantitas positivas directas, negativas inversas, imaginaaaarias laterales nuncupavissemus, pro tricis simplicitas, pro caligine claritas successivet. »

Riemann, in his famous inaugural lecture in 1854 *Ueber die Hypothesen welche der Geometrie zu Grundeliegen*, refers to this work of his master, and imposes the term variety giving to it its generality and a first conceptual though yet incompletly formalized basis.

A new step towards the conceptualization of the general notion of variety was accomplished by Betti in 1874 [4]. As Heegard [29] says :

« Betti has understood that one has to take an analytical basis in order to be able to define a n-dimensional variety… However, the given definition is too vague to allow the accuracy of the consecutive researches. »

Poincaré, in his *Analysis Situs*, gives the first formalization of a variety, in modern language, using submersions and immersions. He begins to define « une *variété à n-p dimensions* » as the counter-image $F^{-1}(0)$ of a submersion $F : \mathbf{R}^n \rightarrow \mathbf{R}^p$. There appears the jacobian of F. Then Poincaré defines analytical varieties by immersions. The principle of the analytic continuation suggests to him to construct some « chaînes continues » of varieties, one being « la *continuation analytique* » of the other ; he is then lead to follow the inverse process which consists in decomposing a variety in partial varieties. From now on, this is the way he will see a variety :

« Considérons une variété V définie à la seconde manière, c'est-à-dire formée d'une chaîne ou d'un réseau de variétés partielles dont chacune est définie elle-même par des relations de la forme (8) et (9). »

He then introduces what we call a change of coordinates, together with its functional determinant Δ. Poincaré considers « un certain réseau continu de variétés partielles » he denotes by (4) :

«dans chacune d'elles, le signe de Δ demeurera constant, mais il pourra changer lorsque l'on passera de l'une à l'autre. Dans ce cas, nous dirons que la variété V est *unilatère.*» « Si, dans la partie commune à deux quelconques des variétés (4), le déterminant Δ est positif, je dirai que la variété est *bilatère*. »

As an example of « variété unilatère », Poincaré of course gives the Möbius band (at that time, this band bears no special name), and, as examples of « variétés bilatères », « toute courbe de dimension 1, toute surface *fermée* à n-1 dimensions. »

Note that a quite close manner to present the definition of orientability is to say that along any closed path on the variety the sign of the determinant of the changes of coordinates is preserved. This is the way various authors choose to define orientability (see for instance Eilenberg-Steenrod [19], p.313) : the fact they use homology groups is not significant in the present case ; what is significant is the existence of an automorphism of the group after the round trip along a closed path, the automorphism being the identity).

A modern course could accept Poincaré's presentation of analytic varieties with the only linguistic change of « unilatère » into « non-orientable » , and « bilatère » (two-sided) into « orientable ».

III. The change of terminology : a long road

III.1 The pertinence of query

A question is why this change of terminology has happened, how and when.

This question concerns the history of a mathematical notion ; so it has to be answered by an historian. Since this question is bound with the way the mind is processing, it is also of interest for psychologists. Since this question is bound with the diffusion of ideas among the scientific community, it is of interest for social scientists. At last, this question may have also a mathematical interest : it might be that the change could be due to mathematical reasons, whatever be the deepness of the reasons ; that is why the question has to be investigated even if one wants to look at history of mathematics from the mathematical point of view only.

III.2 1834-1886 : the silent period

According to the French dictionary, *Le Robert*, the word « orientation » appear for the first time in the French language en 1834. I have not been able to know where, in which field. I have found the use of the terminology in mathematics for the first time in a book by the differential geometer Lamé. In his *Leçons sur les coordonnées curvilignes* [35], published in 1859, he explicitely speaks of « l'orientation du plan tangent. » Since these coordinates, later on called a local parametrisation, play a key rôle in differential geometry, we may conjecture that, given its diffusion along decades, the well-known (at the time) book of Lamé has influenced the terminological use of some authors.

Any way, that was not the case of Jordan. His important work in linear algebra and on polytopes ignores the word orientation.

It has not been the case of Darboux either. But we may suspect an influence of Darboux on Poincaré who associates to a given a variety its « *opposée* », obtained from the initial variety by inverting the order of presentation of any two functions or of any two variables. Darboux, in 1876, publishes an analytical study on the *Surfaces découvertes par Möbius et dans lesquelles on peut passer d'une face à l'autre par un chemin continu* [17] where he shows that the normal to the band in a point can take two opposite directions. Because of his interest in the Möbius band, Poincaré was certainly aware of this article of Darboux whose official position was besides important.

An other geometer had to be checked, E. Laguerre. Let us look at his last paper, published in 1880, *Sur la géométrie de direction* [34] : he only uses the terms « sens » and « direction ». « Laguerre appelle « courbe de direction » une courbe orientée séparable analytiquement de l'enveloppe des droites orientées opposées à ses tangentes orientées. » will write later on Élie Cartan in *La théorie des Groupes continus et la géométrie* [10], a free translation[1] for the *Encyclopédie des Sciences mathématiques* of a G. Fano's article [21] writen for the German corresponding encyclopedia.

III.3 1886-1922 : the decisive period

We have to wait for the end of the years 80 of the last century to attend the more and more frequent use of the term « orientation ». In the course in vectorial analysis written for the students in physical sciences and published by himself in 1881, then in 1884, J.W. Gibbs [26] never uses the term, though this use would have sometimes be convenient to shorten the statement. As soon we shall see from a quotation by Genty, this course might have had some diffusion, the subsequent work of Gibbs as well. The term « oriented » appears under Gibbs'pen only in one of his latest article *On multiple algebra* [27], published in 1886. This article has a true value for who is interested in the history of linear algebra :

[1] Cartan's article appears only in the last volume of his complete works. The printing of this article has been stopped in 1915, whence Fano's article dates from 1907.

« Deferring for the moment the discussion of theses topics in order to follow the course of events, we find in the year following the first *Ausdehnungslehre* a remarkable memoir of Saint-Venant ‡, in which are clearly described the addition both of vectors and oriented areas, the differentiation of these with respects to a scalar quantity, and a multiplication of two vectors and of a vector and an orientes area. These multiplications, called by the author, *geometrical*, are entirely identical with Grassmann's external multiplication of the same quantities. »

At that time, one encounters the term « orientation » but exceptionnally, as for instance in this article by M.G. Kœnigs [33] published in 1887 :

« Soit C un contour de l'espace formé d'une courbe analytique ou d'un nombre fini de courbes analytiques ; soit aussi un point donné O dans l'espace. Je fixe un sens de parcours sur le contour ; alors toute portion du contour possèdera, par cela seul, un sens de parcours parfaitement défini, et la corde qui fermera cette portion de contour, ou en sera la somme géométrique, ou en sera un segment déterminé non seulement en grandeur et orientation, mais même en direction. »

In 1898, Paul Heegaard[2] publishes his thesis in the field we call today differential topology ; it contains a constructive criticism of Poincaré, and is at the origin of the second Poincaré's article on *Analysis Situs.* The original work was writen in Danish ; the French Mathematical Society published a translation in 1916 [29]. The paragraph X of the memoir is devoted to the « Variétés orientées » : this terminology seems to have been introduced for the first time in the context of varieties. We can read in « L'indicatrice » :

« Le voisinage d'un point sur une courbe est un petit arc dont nous désignerons les extrémités par 1 et 2, de sorte que 1, 2 indique un sens positif sur l'arc ; nous appellerons 12 une indicatrice de premier ordre (cf. Dyck, *Math. Ann.*, Bd 32, p. 473) et nous dirons qu'une courbe munie d'une indicatrice est orientée. »

Let us recall that the indicatrix in a point of differentiable m-manifold can be represented by an oriented sphere S^{m-1}. In the subsequent paragraph, Heegaard considers (in our modern language) a basis of the Euclidian vector space $\mathbf{R}^m$. If b^i is a vector of the basis , he calls « demi-ligne » the half-line λb^i where λ is a non negative real number. On each such half-line he takes one point ; he denotes by (a), (b), … these points : « il est supposé que les m points aient une disposition générale telle que le déterminant $\Delta = |a, b, \ldots, k|$ soit différent de zéro. » Then he adds several lines later :

[2] I would like to thank Mogens Niss (from Roskilde University) for having looked for Heegaard's original thesis and for having send a copy.

« Les m variétés planes, avec les demi-lignes, forment ce que nous appellerons un *coin* à m dimensions ayant (o) pour sommet et les demi-lignes pour arêtes ; nous l'appellerons *orienté* si l'ordre dans lequel les arêtes est déterminé, … »

Considering the points of intersection (a), (b), … of the edges of the corner with the indicatrix sphere of radius dr, Heegaard verifies that his determinant Δ est Poincaré's functional determinant. Thus he concludes to the equivalence of his theory through the indicatrix with Poincaré's one.

In the chapter XII of his memoir, Heegaard brings out the personnal character of his terminological choice :

« La définition du nombre de connexion *p* dépend et de la manière dont on définit les variétés V_1, V_2, … mentionnées, et de l'importance accordée aux mots : « former frontière ». V_1, V_2 doivent être, entre autres, *bilatères* ; de plus, elles doivent être ce que nous appelons *orientées*, et il est exigé que ces orientations *concordent* (comme nous disons) avec l'orientation de la variété limitée par elles (P. et S., Chap. II, n° 8). »[3]

We are going to see that many years, a quarter of century at least, will elapse before the term orientation be accepted in differential topology. But its use in geometry will get common very soon.

Let us before quote the text by Genty [25], published en 1903, that we anounced when mentioning Gibbs'work :

« Nous emploierons, dans ce qui va suivre, les notations de M. Gibbs, savoir : α, β, γ, ... étant des vecteurs, α.β désignera Sαβ, c'est-à-dire le *produit projectif* des vecteurs α et β, et α × β désignera Vαβ, c'est-à-dire le *moment orienté* de ces deux vecteurs. »

We shall notice here that the expression « moment orienté » is taken from mechanics. More generally, the influence of physics on the terminology will appear through the works of the German mathematicians.

We owe to Study the systematic use of the terms bound to the verb « to orient », and the first studies on the orientation in the geometrical spaces. His first article published in 1901, *Die element 2Ordnung in der Ebenen projective Geometrie* [45], will be analyzed by G. Fano in his above-mentioned article for the German encyclopedia. Fano is respectful to the terminology used by the authors of whom he states the works. The paragraph 16 of his text is titled *Studys Geometrie der Element 2.Ordnung in der Ebene*, translated by Cartan

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[3] Here is the original text by Heegaard : « Definition af Sammenhængstallet *p* afhænger baade af, hvorledes man definirer de Mnagfoldigheder V_1, V_2, …, som omtales i den, og af, vvilken Betydning man tillægger Ordet « begrænse » (former frontière). V_1, V_2, … skulle blandt andet være *bilaterale* og være, hvad vi kalde *orienterede*, od der folanges, at disse Orienteringer med vor Udtryksmaade skulle være i *Overensstemmesle* med Orienteringen af den Mangfoldighed, de begrænse (P & S. ; 2 det Kap. No. 8). »

under the name *Géométrie projective des éléments du 2nd ordre du plan*. We can view it as a geometric study of the jets of seond order of analytical curves. Here are some lines by Fano and their translation by Cartan which have some interest for us :

« Trennt man di drei Bestimmungsweisen dieser Einheitswurzel voneinander, so ergibt sich die analytische Darstellung eines « orientierten » Elementes 2.Ordnung. Von drei Oirentierungen eines reellen Elementes ist auch nur eine reell ».

« Si l'on se donne une des trois déterminations de cette racine cubique, on définit analytiquement un élément orienté du second ordre. A un élément du second ordre (x, y, y', y") donné correspndent donc trois éléments de ce second ordre orientés ; si x, y, y', y" sont réels, un seul de ces éléments orientés est réel. »

The reading of E.Cartan's works shows that it is essentially through the German authors and Fano that he will use the term « orienté », besides in a much more systematic way. For instance the title of the paragraph 13 of Fano's article, *Die Liesche Kugelgeometrie*, becomes, under Cartan's pen, *La géométrie des sphères orientées de Lie.*

In 1903, Study publishes his book titled *Geometry der Dynamen*, then in 1907, in the American Journal of Mathematics, an article devoted to the non-Euclidian geometries [47] where he frequently uses the verb « to orient », as for instance in the following lines :

« denken wir uns die erster Ebene *orientirt*, d.h. denken wir uns etwa ein in ihr gelegenes Dreieck mit einem bestimmten Umlaufssim versehen (womit zugleich einer jeden Normale des Ebene eine bestimmte Richtung zugeornert wird), so damit auch die zweite Ebene orientirt, durch den entsprechenden Umlaufssinn des entsprechenden Dreiecks. »

In his following works, he will use this terminology more and more frequently. In an article published in 1913 [47], he first sets up an inventory of the scientific domains where the notions of rigth-left, sense play a determinant rôle. Taking his inspiration from Study's works, W. Blaschke publishes in 1910 an article [5] where the adjective « orienterten » appears in the title, and where he introduces the « *orientierte Normale* ».

Nevertheless, the use of the term orientation is not yet spread over all the branches of geometry. In 1915, J.W. Alexander, *A proof of the invariance of certain constants of analysis situs* [1], still speaks of « sensed cycles ». However, later in the same volume where appears Alexander's paper, on can read under D.A. Barrow's pen [3] :

A wheel mounted on a spindle may be rotated in either two directions. Thus rotation is a property which may be adjoined to a wheel in eithe two ways. Analogously, *orientation* may be thought as an arbitrary property which may be adjoined to any non null circle in two ways, thus giveing rise to two *oriented cycles.* In the mathematical development of this idea … »

In 1920 is published an article by Tietze [48] who sets up a list of mathematical domains where the implicit or explicit notion of orientation appears. Though it does not appear in the title, the use of the term orientation is constant in the article. References to Study are numerous. J.W. Alexander reads this article by Tietze.

In 1922, Veblen, in his *Analysis Situs* [49], sets up the modern terminology :

« A manifold is said to be *orientable* or *non-orientable* according as the complex defining it is or is not orientable. »

He reports on the same page, 71, that

The term “orientable ” was suggested by J.W. Alexander as preferable to “two-sided” because the later connotes the separation of a three-dimensional manifolds into two parts, the two “sides”, by the two-dimensional manifold, whereas the property we are dealing with is an internal property of the two-dimensional manifolds*.

This book, where Heegaard’s memoir figures in the bibliography, followن in 1932 by Veblen and Withehead’s book [51] in 1932, has imposed the use of the term « orientation » on the American school.

Through algebraic topology, this school has made a thorough study of the bounds between the orientation of combinatorial complexes and the topological orientation, and some studies on relative orientation. The history of these progress can be found in the Lefschetz ’s classical book on *Algebraic topology* [37]. The last significant progress was made in 1944 when Eilenberg proved [19] that an oriented simplex is the same as an ordered simplex. It must be said that the intimate relation between orientation and order has never left the thought of all the former mathematicians involved with these questions.

III.4 The case of the French school

In the first part of this century until the second world war, with a few exceptions like the blind mathematician Antoine with his famous collar, the French school has not been involved in differential topology. Differential geometry was under the main influence of É. Cartan. Some of his students at that time told me to have felt some inconvenience in Cartan’s lectures because of a too strong appeal to intuition.

In preference to Heegaard’s terminology (oriented varieties), the definitions and the terminologies used by Poincaré (opposed, one-sided, two-sided varieties) will be kept in France until at least 1928. As an example, Lefschetz ‘s monography [36], published in 1949 but « basée, en partie, sur une série de conférences faites à Rome, au printemps 1921 », takes these various denominations into account, giving the main position to Poincaré's ones.

In the twenties, the use of « orientation » appears but timidly. In his *Leçons de Géométrie vectorielle* [6] published in 1924, we can find the term two times : one time about the « normale orientée » of the Möbius band, a second time, at the beginning of the book (p. 19), about the geometrical meaning of the determinant :

> « Si l'on conserve la variation précédente, l'orientation du trièdre OABC, le signe du volume (OA, OB, OC) et celui du déterminant correspondant demeurent invariables. Le déterminant est positif ou négatif suivant que le trièdre OABC et le trièdre $OA_oB_oC_o$ sont de même sens ou de sens contraire. »

Here, the term « orientation » is taken in its intuitive meaning and has not been defined. In Élie Cartan's memoir writen in 1925, *La Géométrie des Espaces de Riemann* [11], the explicit notion of orientation yet does not appear. In 1928, Cartan takes again his memoir under the name of *Leçons sur la Géométrie des Espaces de Riemann* [13] ; each time it is necessary, he enriches it with references to the notion of orientation, but without any definition :

> « deux trivecteurs sont dits égaux si leurs triplans sont parallèles, si les parllélépipèdes construits sur les trois vecteurs transportés en un même point ont même volume, si enfin l'orientation de ces deux parallélépipèdes est la même. » (p.12)

One can recognize here Bouligand's presentation. In the same book, É. Cartan still writes :

> « 50. La notion générale de variété est assez difficile à définir avec précision. »

After the preparatory works by Poincaré (1895), Veblen and Whitehead (1932), the masterly study of H. Whitney [52] in 1936 will answer the Cartan's questioning.

The geometrical frame and the terminology are now fixed for the years to come. The way to present the topological orientation in the fiber spaces is not followed with any conceptual adding.

IV. The notion of geometrical orientation

IV.1 Geometrical orientation according to Élie Cartan

Élie Cartan begins his 1941's lecture by observing that

« Le terme d'"orientation " intervient souvent en géométrie et dans des sens assez variés. »

For instance, in the paragraph 13 quoted above on « La géométrie des pshères orientées de Lie », he gives the definition of he orientation of the sphere in projective geometry :

« Nous conviendrons de dire que l'ensemble des six paramètres $x_1, \ldots, x_6$, liés par la relation $\Phi = x_6 - \Omega(x_1, \ldots, x_6) = 0$ définit une *sphère orientée* de rayon $r = x_6 / \Sigma c_i x_i$. Une sphère orientée est donc une sphère définie sans ambiguité de signe. »

The article refers to the first works by Lie [38], [39], who uses the term « Richtung ». In 1947, É. Cartan [15] will give a close definition of the orientation of a sphere :

« orienter une hypersphère proprement réelle, c'est choisir une des deux régions dans lesquelles l'hypersphère sépare l'espace anallagmatique. »

Let us return to the text of Cartan's conference. He gives rapidly the following trivial example : take a point on a line, and make a π-turn of the line around the point ; the source and its image coincide. If the line has a given orientation, the source and its image have opposite orientations. A family F of non oriented « figures » form a « corps » (field) Cartan says, if a Lie group G acts on F transitively so that any figure of the family belongs to the family. Cartan introduces a two-valued orientation on F. It allows to define two counter-images of F, (F, -) and (F, +). On each leave (F, -) or F, +) of the covering acts a closed subgroup of G, thus a Lie group again. Any oriented figure, (f, -) or (f, +), has a subgroup of stability (or isotropy group) $H_{(f,-)}$ or $H_{(f,+)}$. Then one immedialy gets Cartan's theorem according to F is orientable if and only the subgroup of stability $H_f = H$ (in G) of f (in F) has two distinct connected components, images of $H_{(f,-)}$ and $H_{(f,+)}$ by projection in G. Cartan works on this observation in several contexts of the standard geometry. Geometrical orientation does not imply topological orientation and conversely, though they can appear simultaneously. Cartan notices that this definition can be extended to the case where the subgroup of stability has p components, giving birth to p distinct orientations. He then says :

« La définition concrète de ces orientations est plus ou moins intuitive, plus ou moins simple, plus ou moins facile, mais on en conçoit la possibilité. »

Cartan's statement, yet not very formalized, has been one of the incentives which have lead R.W. Sharpe to define an other notion of geometrical orientation. Except this important fact, the conference is remained fruitless until now.

IV.2 Geometrical orientation according to Sharpe

Owing to a good use of the « modern » formalism, this author has successfully ligthened some zones, still in the shadow, of the geometrical scenery observed and described by Élie Cartan. For instance, he takes again the definition of the local representation of a manifold on an osculator space or on his tangent space, and the notion of development (Lagrange, Lie) of the manifold along a path (cf for instance [13], p. 105, 168 or 178). The use and the analysis of this development have plaid an essential part in Cartan's work.

Let us recall that a topologically orientable (differentiable) manifold M is characterized by the fact that the determinant of the jacobian of the changes of coordinates has a constant sign along closed paths. In other respects, if two distinct paths join two points on M, the development of these paths on the tangent space leads to two paths with the same origin but whose final ends do not coincide *a priori*. Sharpe takes an hint from these two facts : he will consider as geometrically oriented a differentiable manifold for which the development insures a form of conservation of the homotopy of the paths. The point of view being new, let us a little bit detail the construction.

Let $f : M \rightarrow G$ be a « representation » of M in the Lie group G : f is simply here a differentiable (i.e. C^∞) mapping. e denotes the neutral element. The tangent fiber space of G, TG, is projected on $T_eG = \mathfrak{g}$ by the Maurer-Cartan 1-form ω_G, which is defined at each v of T_gG by the relation $\omega_G(v) = L_{g^{-1}*}(v)$, where $L_{g^{-1}*} : T_gG \rightarrow T_eG$ is the linear tangent mapping of the left translation by g^{-1}. The counter-image of this form in TM, $\omega = \omega_f = f^*(\omega_G)$, is the Darboux derivative of f, so that f can be viewed as a « primitive » of ω. When M = I = [a, b], f defines the *development* of ω in G along I, with origin f(a). If $\sigma : I \rightarrow M$ is a path on M, the composition $f \circ \sigma$ defines the development of ω along σ with origin $f(\sigma(a))$. It happens that, for such a Darboux derivative, the development of two homotopic paths on M have the same final ends in G.

We know that we can represent (here in the classical sense of group theory) G in the linear group $Gl(\mathfrak{g})$ of the vector space $\mathfrak{g}$ by the adjoint mapping or action Ad, where Ad(g) is the change of frame defined by $Ad(g)(h) = ghg^{-1}$. Thus one can also define the development of Ad(ω) along σ in$Gl(\mathfrak{g})$.

What is today often said to be an *homogeneous space* was formely called a *Klein space.* We may also call it a *principal geometry*, even a *classical geometry.* It consists in a given quotient-group M = G/H where G is a Lie group, and H a closed subgroup (thus a Lie subgroup) of G. G is the principal group of the geometry (the *Hauptgruppe* according to Klein). The Maurer-Cartan form has specific properties. These geometries have been generalized into principal bundles. Sharpe calls « *Cartan geometry* modelled on $(\mathfrak{g}, \mathfrak{h})$ » or on (G,H), a fiber bundle on M, with structural group H, Lie subgroup of G, for which the Maurer-

Cartan form ω verifies the properties of a form associated with a classical geometry.

Then let $\xi = (P, \omega)$ be a Cartan geometry on M, x a point of M and p a point of P above x. The element h of H *preserves the geometrical orientation* with respects to the point p if there exists a path $\lambda : (I, a) \to (P, p)$, going from p to ph, whose development $\tilde{\lambda} : (I, a) \to (Gl(\mathfrak{g}), e)$ joigns the identity e to Ad(h).

The set of h which preserves this geometrical orientation is denoted by H_{or}. When this set is H, the geometry is said *geometrically oriented*. The fact that a classical geometry, whose principal group G is connected, is geometrically oriented in Sharpe's sense plays an important rôle.

V. Some historical complements

V.1 Orientation : the awareness of the notion

I have been rather astonished by the fact that, until the end of the XVIII-th century, the notion of orientation has been absent from all the philosophical and mathematical considerations : was the notion was too much secondary to deserve an immediate attention, or too deep to be easily accessible ?

The notion of sense, of direction is however a universal notion since we can observe its impregnation in all the reigns of nature. We are more or less familiar with the feeling of the flow of the time, with the sense of rotation of the planets, with the orientation of motions induced by the presence of potentials and fields,with the spin of elementary particles whose significance remains unclear, with the more obscure chirality of the universe.

The need to reach the sources of energy to insure stability and development, to avoid and sometimes destroy the predators has intimately modelled the biological world with the notion of direction. Gradients of pressure, of light, of chemical emanations have given rise to the birth of the sensory placodes : they differentiated to allow the analysis of the various sensorial fields into three main components.

The common language now makes use of « orientation » in political sciences, in economics (« the orientation of the market »), or even in philosophy : for instance *Orientation philosophique* is the title of a book by a French philosopher, Marcel Conche.

The question arises of the possibility to formalize of the notion of orientation. The question was implicitly set up by Emmanuel Kant :

« L'incitation initiale de Leibniz à approfondir l'étude des relations spatiales conduisit Kant, qui connaissait déjà les opinions d'Euler et de Buffon sur ce sujet, à examiner à son tour la question. Dans son étude intitulée *Des premiers fondements dela différence des régions de l'espace* (1768) [31], Kant soulève de manière implicite les questions d'orientation, à travers l'examen qu'il entreprend des problèmes de symétrie et de congruence. Son propos, quoique remarquable, paraît ici trop philosophique pour avoir une influence durable chez les mathématiciens. Toutefois, une autre publication, *Que signifie s'orienter dans la pensée ?* (Octobre 1768), doit être mentionnée : traduit-elle la présence dans le monde scientifique de son époque d'un courant de réflexion centré sur l'orientation, ou bien ce texte est-il l'expression de la réflexion originale de Kant ? Par qui cette publication a-t-elle été et sera-t-elle lue ? » [7].

Let us here quote, in our modern language, two Kant'sentences which have some interest for us – Kant has underlined the words he thought to be important :

«To *orient oneself* means in the proper sens of the word : from a given region of the sky (we divide the horizon into four regions) find the ohers, particularly the *Levant*. ….

I can now enlarge the geographical concept of this process of orientation and understand it as follows : to orient oneself in general in a given space, and thus in a simply *mathematical* way » [32].

As we know now, more than a century will be necessary to elaborate a first mathematical notion of orientation.

V.2 The Carnot-Grassmann period

Two centuries ago, in 1797, Lazare Carnot publishes his *Réflexions sur la métaphysique du calcul infinitésimal* [8]. The end of this book prefigures his next book, *Géométrie de position à l'usage de ceux qui se destinent à mesurer les terrains* (1803) [9]. He writes in the *Réflexions* (in a Note relative to the paragraph 162) :

« 3. Avancer qu'une quantité négative isolée est moindre que 0, c'est couvrir la science des mathématiques, qui doit être celle de l'évidence, d'un nuage impénétrable …

… si l'on me donne pour exemple de quantités opposées un mouvement vers l'orient et un mouvement vers l'occident, ou un mouvement vers le nord et un mouvement vers le sud, je demanderai ce qu'est un mouvement vers le nord-est, vers le nord-ouest, vers le sud-sud-ouest, etc ., et de quels signes ces quantités devront être affectées dans le calcul ? »

Géométrie de position is a fascinating book, annoucing many directions of deep developments of mathematics ; some of them are described in [7]. We shall meet some others in this paragraph. A great part of the book is devoted to attempts at explanation of negative numbers. As we have already hinted in the above quotation, he introduces implicitly the notion of orientation when he speaks of the « sens » of a move on a line or on an arc to make understable why some numbers, like the cosinus of an angle, can be positive or negative.

Given the personnality of its author, its title and its content, it is highly probable that this book has influenced many mathematicians in the early nineteenth century.

Argand [2], in his *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (1806), introduces, concerning the half-line or « ray », the term of « direction » :

« On les appellera *lignes en direction* ou, plus simplement, *lignes dirigées.* Elles seront ainsi distinguées des lignes *absolues*, dans lesquelles on ne considère que la longueur, sans aucun égard à la direction (*). »

Here is the note (*) :

« (*) L'expression de *lignes de direction* n'est qu'une abréviation de cette phrase : *lignes considérées comme appartenant à une certaine direction.* Cette remarque indique qu'on ne prétend point fonder de nouvelles dénominations, mais qu'on emploie cette façon de s'exprimer soit pour éviter la confusion, soit pour abréger le discours. »

Let us notice that these lines are preceded by a mechanical illustration of the difference between $\overline{\mathrm{KA}}$ and $\overline{\mathrm{AK}}$, represented by parallel forces to a same line, but « en sens contraires ». Directly influenced by L. Carnot and by Argand, J.-F. Français publishes, in 1813, in the *Annales de Mathématiques de Gergonne*, an article with the same ideas, where he introduces the « droites positives » and « négatives ». None of these mathematicians, neither Gergonne nor Servois, will ask the question whether it is possible to « direct » a plane surface, or, *a fortiori*, the usual space.

In the introduction (*Dissertation préliminaire*) to the *Géométrie de position*, L. Carnot writes :

« Mais dans cette section (la quatrième), j'examine séparément d'une part, les rapports qui existent entre les angles seuls, et de l'autre, ceux qui existent entre les lignes seules : j'y donne la notion du centre des moyennes distances. Je remarque que ce point est le même que celui qu'on nomme en mécanique, *centre de gravité*. D'où je conclus que ce centre appartient à la géométrie, et qu'il seroit très-avantageux pour les progrès de cette science, de rétablir l'ordre naturel des idées. »

Möbius, may be inspired by Carnot, has developped this idea in his memoir on *Der barycentrische Calcul* (1827). He recalls the work by Meister, published in 1769 in the *Novi Commentari Gotting.*, where negative areas appear. This memoir is quoted by Gauss in a letter directed to Olbers, on the 30th of October 1825. Gauss[4] writes :

[4] In his book [18], P.Dombrowski, by the personnal use of the term « orienté », gives free translations of various passages of some Gauss' letters.

> « In so fern man nemlich geometrishce Relationen analytisch behandelt, hat man zwar längst Linien von positivem und negativem Werth recht wohl verstanden, und eingesehen, dass dabei immer explicite oder implicite ein gewisser Sinn (sens, Richtung) zum Grunde liege, nach welcher der Linien als waschsend angesehen werden act. »

In 1837, in his *Lehrbuch des Statik* [41], Möbius undertakes the study of polyedrons, as a generalization of the polygons met in statical mechanics : he introduces the move along the edges of a polyedron.

Gauss, in a manuscript [22] writen about 1840, takes also into account the sense of the displacement in a kind of cellular decomposition of the plane representing the complex line.

But we remain in presence of a local notion of orientation, relative to the edge ; we do not reach a more general notion, more global, that would concern the polyedron or the cellular complex.

The same remark applies to Grassmann'works. The « sense » of the edges and the order in ther multiplications have an important place (cf for instance his 1855 article *Sur les différents genres de multiplication* [28]), but the term and especially the global notion of orientation do not appear in his writings.

The Malus'researches on the polarization of light (cf his *Optique* [40] published in 1808 in the XIV Cahier de l'Ecole Polytechnique), thoses of Oersted and Ampère (cf his first *Mémoire* in 1820) on electromagnetism have shown the physical importance of the directional facts, of the notion of closed oriented path. But these are oriented in an implicit manner, and arrows never appear in their drawings.

Physicists and mathematicians of this first half of the XIXth century do not reach in clearing a mathematical notion of orientation in itself.

VI. Mathematical comments : towards extensions of the definitions

VI.1 OH-orientation

What kind of notion is the first notion of orientation : say algebraic, topological, or something else ?

Let us first recall that, until now, orientation has been mainly used in geometry, finite or not, in differential topology, and in differential analysis through the Lie derivative. In these fields, there can be local orientations. One problem was to define a notion of global orientation ; one property of such a global notion is its compatibility with some or any local orientations of sub-objects.

What was considered first were one-dimensional objects. In that case, to set up a definition has been quite easy ; orientation can be viewed as a way to define linear order. It took a very long time to prove that topological orientation is equivalent to the absence of obstruction to some underlying order : as noted before, this was achieved in 1944 by Eilenberg.

The first extension to two-dimensional objects has been made through local parametrisations by two one-dimensional objects in general position. If the two-dimensional object is a connected manifold (surface), one can choose these parametrisations in such a way that a one-dimensional sub-object is a connected line. A possible obstruction or incompatibility appeared when this line is closed : there may be two opposite normals in a point, or equivalently, the sign of the determinant of the motion along the closed path takes two opposite values.

This extension was made in the intuitive usual space, to get a codimension 1 submanifold of $\mathbf{R}^3$. The Möbius and Poincaré's terminology refers to this situation. Alexander and Veblen, under the influence of the geometers, but also quite familiar with the plurality of dimensions, convinced themselves to use a more general terminology because of the intrinsec character of the definition of topological orientability. According to the point of view, this reason may be considered as deep, or not because it is natural in mathematics to look for and to emphasize the natural and the universal characters of concepts and notions.

Now take a codimension k submanifold in some space, with a k-dimensional normal space. The definition of orientation can be refined according to the possible change or not of the sign of the determinant along a bundle of r closed paths belonging to a set of local parametrizations. Since these closed paths can locally form braids of various types, one can see that a lot of refinements can occur.

Topological r-orientation is a constitutive character of the manifold. It refers to an internal property. It is a property bound with the possible presence of *a specific obstruction* O associated with *a specified generalized homotopy* H, given rise to what we may call OH-*orientation*.

It is easy to think to such an obstruction given by a metric character : take a car which has to follow a closed path with various kinds of up and down slopes. Given the non-linearity of the oil-consumption, the integral O evaluating the cost of the round trip will be in general different according to the sense of the motion along the path H. In that case we shall consider that the cost of a round trip is an obstruction to put on the same step a motion on the path in one sense or in the other, and we shall say that the manifold is (at least locally) OH-non orientable. If, whatever be the closed path and the sense of the displacement, the cost of the round trip is the same, we shall say that the manifold is OH-orientable.

Sharpe's orientation comes from the same philosophy he uses not with the manifold itself, but with a specified representation of the manifold.

The denomination « OH-orientation » reflects his double character. The H refers to more intrinsic topological properties than the O : the ribbon and the Möbius band have the same band as underlying topological space. They differ from the process of identification of a part of the boundary of the band. To get the Möbius band we need to make a twist of the band, we add a supplementary property to the original band which needs an extra spend of energy, and for which we have to pay : the O has a metrical character.

We may thus conceive this OH-orientation as a fine tool which helps to describe the structure of topological and metrical manifolds.

VII. 2 Cartan's geometrical orientability and OH-orientability

Before taking up some mathematical considerations, let us again notice the long period, about 30 years, which separates the time of translation of Fano's article on geometry from the conception of the geometrical orientability. Let us recall that during almost all this period, Cartan has used the term orientation in an intuitive manner. Here are some souvenirs given by P. Lelong[5] :

> « Assidu à ses cours, j'ai bénéficié de son admirable enseignement qui nous ouvrait des horizons noouveaux, mais je me souviens aussi combien, en invoquant l'orientation, on faisait appel à l'intuition. Mais, tout comme chez H. Poincaré, on était assuré que, en agissant ainsi, on ne faisait qu'éviter des développements secondaires, pour aboutir à des équations précises, appliquées et vérifiées dans un vaste domaine de recherche. On admettait alors que l'appel à l'intuition fût un raccourci emprunté pour éviter un trajet trop long. »

Let us recall that Élie Cartan was a differential geometer, under the strong influence of Erlangen's program. He was the best specialist of Lie groups. In his time, internal differential topology was immature, and topological orientation did not play a great rôle in the studies. It is not unreasonable to suggest to that Cartan was induced to think about orientation by the fact that topological orientation was taking a significant position in differential topology.

That he takes up the problem using algebraic methods *à la* Klein will surprise nobody. Here, the use of closed paths and H(omotopy) conditions appear through the connexity of the isotropic subgroups. With Cartan's method, the obstruction O to connexity creates the geometrical orientability. In some sense, Cartan's algebraic approach is dual from the previous one.

The question of relative orientations of sub-objects has not been completly worked out in differential topology – in spite of some general statements arising from the combinatorial treatment of manifolds (cf [37] and [20]). On the contrary, and for instance, Cartan's approach allows to give immediatly effective results on the relative geometrical orientation of the flag of linear subspaces in Euclidian or in projective geometry.

The non coincidence, in general, of geometrical and topological orientation is mainly due to the fact that the groups involved in geometrical orientation usually operate on very large spaces, and are different according to the space involved, while topological orientation has both a local and global

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[5] In a letter about a French version of this paper. The content of this letter and a telephonic talk have been stimulating and have led me to change the order in the original paper and to make some addings. I would like here to thank him.

character, concerns smooth spaces with very homogeneous properties, so that the local diffeomorphism groups are all the same. But then one can expect that Cartan's orientation, when first applied to the elementary case of simplicial complexes, will lead to the same results as the effective use of topological r-orientation.

VII Conclusion

Orientation is a recent concept, for which, yet, no general and definitive axiomatization has been proposed. It is not a primary concept like the familiar ones of number, or of point, or of line. The best unformal definition we can give of non-orientability might be the presence of an obstruction to some ordering.

The notion of order is formalized ; the notion of general obstruction is not. It is something that prevents from. As for the term orientation, we can find precise *uses* of the term obstruction in the frame of algebraic topology. But obstruction can appear in other contexts : singularities, poles might be felt as sources of obstruction.

May be, we lack of mathematical experience to be able to achieve a satisfying formal definition of such terms. But when such a formal statement is given, it restricts sometimes considerably the semantic field attached to the terms. The fact that these terms do have a large semantic field, not restricted to mathematics, is certainly a good sign of their pertinence.

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