DYNAMICAL SYSTEMS
A RENEWAL OF MECHANISM

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BIFURCATION AND CONTINUITY

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The study of bifurcation is a recent one, and we may ask why it has been necessary to wait lately to undertake the acute taxonomic study of separation phenomena: insufficient observation due to incomplete equipment, very rapid transformation during the phases of change, instability of some transitory states, absence of conceptual tools inciting the search and the observation of change.

Newton with his fluxion, Poincaré with his qualitative study of movements in the plane, forged the conceptual tool we use today, and of which we can celebrate the centenial. The series of Poincaré'articles, on the one hand **Sur les courbes définies par des équations différentielles,** on the other hand **Sur l'équilibre d'une masse fluide animée d'un mouvement de rotation,** were published from the years 1881 to 1886 for the first, during the year 1885 for the second; we may reasonably think that Poincaré has been working on this last article since 1884. In the first series, Poincaré practices bifurcation without saying it, including the one today inexactly attributed to E.Hopf. In the second series, Poincaré introduces the expression forme de bifurcation. Poincaré does not borrow this terminology from the physicists Thomson and Tait who inspired Poincaré's study by looking at the various types of behaviour of a fluid in rotation. It is very likely that he uses a railway image, as he will explain later when introducing the term trifurcation, when he studies the geodesics on a surface and their singularities. It is apparently the first time that the term "bifurcation" appears in mathematics, whereas it has been used in French since 1560, concerning anatomy.

What does Poincaré mean by forme de bifurcation ? He considers several examples, in particular the two following, surprisingly familiar to an amateur of elementary catastrophies. Poincaré looks upon two fonctions de forces:

$$F = Ax_1^2 + \frac{1}{3}x_2^3 - y^2x_2 - \alpha yx_2^2$$

and

$$F = Ax_1^2 + \frac{1}{4}x_2^4 - y^3x_2.$$

He wonders about the evolution of the positions of equilibrium of the systems defined by the functions F according to the value of the parameter y. He is interested in the manifold of the positions of equilibrium in the (x,y) space, and in the points where several singular points joign together. In these points, the Hessian denoted by Δ vanishes, and in their neighbourhood, it may happen that the variety of equilibrium points splits into several branches. Let us take the first example with $\alpha = 0$. The singular points are defined by $x_1 = 0$, $x_2 = \pm y$: the variety of equilibrium points consists of two lines. In other respects $\bar{\Delta} = 4y$ is null for $y = 0$. Poincaré says that the origin belongs to a forme de de bifurcation, since in its neighbourhood, two distinct formes d'équilibre can be found. This phenomenon happens each time the sign of the Hessian changes.

Whatever is the importance of his works, Poincaré did not feel the need to define the notion of bifurcation in the modern way. None of his subsequent works contain an allusion to the notion: Poincaré did not see its importance.

Without doubt Andronov must be credited with the creation of the theory when, with Pontriaguine, in 1937, he introduced the notion of structural stability. The most general definition of a value of bifurcation of a parameter includes deterministic and non-deterministic cases. True or not, our deterministic conception of the world incites us to keep a deterministic definition of such a value, for which the flow of a differential system looses its structural stability.

Passing through a value of bifurcation, the dynamic behaviour of the system degenerates; in its neighbourhood, this behaviour is in general different according to the branch which is considered. These properties have a physical significance of such importance that any mathematician who would enter the field of study of bifurcation should not be discouraged a priori. Change characterizes life, and this change is fulfilled with the appearence or not of novelties i.e. of new dynamic behaviour and morphology, these being able to be represented by temporal sections of adequate subsets of trajectories. It may happen that the bifurcation be accompanied by a dynamic behaviour unknown until now, thus with an endogeneous novelty. That is all the more plausible as, even for very primitive systems like those which can be simply described by two state variables, the modes of bifurcation are extremely numerous.

The study of bifurcation gives rise to three orders of problems, which are linked: the first is bounded to the necessary inventory of the forms of bifurcation; the second concerns the modelisation of phenomena; the third refers to the status of bifurcation.

The catalogue of bifurcation is set up according to three usual processes that it would be also artificial to totally disjoint: the first is physical the second geometrical, the third is analytical.

The physical approach consists in observing, in the field or by reflexion, the modalities and the forms of the flows. The interest of these is justified by the manner in which the physical universe appears, stratified, foliated in tiny streams of variable dimensions, and in perpetual state of flux. In particular, the tiny streams of the flows of usual liquids materialize well the trajectories and, according to the properties of the fluids or of the surfaces, offer to the observer a lot of varied examples of bifurcation. As we have just seen, Poincaré introduced the notion of bifurcation following an experimental study by Thomson and Tait. The experimental studies which are accomplished nowadays, are also used as sources, supports and justifications of searches on bifurcation.

Poincaré 's work gives evidence of the advantageous use that can be made of geographic illustration. However, the physical approach of phenomena does not seem to offer exhaustive procedures of examination and classification. Recourse to mathematical methods is necessary to succeed in recognizing the multiplicity of forms of surface possible on which we want to observe flows. But even in this chapter, our knowledge remains insufficient as soon as we want to take metrical data into account in the classification, whatever the scope of old (Monge Poincaré) or recent works (Sovietic school, Ozawa) may be.

By their implicit reference to the physical reality, the geometrical methods used in the finding out of bifurcations are scarcely distinguishable from the physical methods. On the other hand, we can consider as more specifically geometric some procedures of examination and of proof: these aim at testing the stability of trajectories. Two of these deserved to be mentionned. The first one, the method of the loop, goes back to Euler, via Cauchy and Kronecker: it concerns periodic or quasi-periodic situations. Poincaré used it in various ways, generalizing Kronecker's index, and defining the first return map and the fundamental group. The second procedure, with respect to a state of reference, associates a geometrical representation under, for instance, the form of a sub-manifold V_r: with respect to this particular situation, the standard situation is stable if it can be represented by a sub-manifold V_s transverse i.e. non tangent to V_r. A picture shows immediatly that a state of transversality is stable as any small perturbation of the parameters keeps this transversality on. On the contrary a state of non-transversality is kept or not by the perturbation. Non-transversality in general chaaracterizes a state of bifurcation.

As soon as the multiplicity of cases become larger, the use of procedures of classification becomes necessary, and the analytical procedures sometimes supply the geometrical procedures. Very rapidly, the study of bifurcation comes up three orders of difficulties: the first one concerns the multiplicity of singular points, of their state of degeneration, of parameters, which in the general case prevents us from having an exhaustive and global vision of the local bifurcations. The second order of difficulties refer to the degenerates cases: for instance the determination of the nature of the singular point, that of its stability are not algebraïc problems. The third order of difficulties comes from the non-recognition in general of the presence of cycles, more generally of invariant tori.

In order to clear away this lack of knowledge, we can think of setting up controlled changes of scale, and thus to bring the presence of tori inside accessible domains, as for instance neighbourhoods of singular points. In fact, the inverse technique is worked out and used by the specialists of non-standard-analysis. Owing to this technique, it is possible to demonstrate mechanisms of bifurcation very close to those one meets in the physical world. We have shown for instance that he may happen that, separated by a zone of turbulence, two cycles travelled on at different speeds, vanish when a part of the first gets very near a part of the second cycle. An intermediate situation may occur when only one cycle vanishes. Experiments could be tried with two liquids of different densities, flowing apart on an appropriate surface of revolution, rotating around its axis. These results show that the geometrical non-transversality is an extreme condition that does not necessarily have to be globally fulfilled in order to get a bifurcation. Such refinements highten our difficulties. To be more complete, it must be added that bifurcation is defined with respect to a certain order in the degree of stability, order equally variable up to infinity.

We understand that, confronted with the huge diversity of cases, the mathematicians intend to explore the universe of bifurcations with more caution. For instance, they limit the study of the stability of behaviour to some subsets of trajectories, or to some sub-manifold of observables. This orientation is quite realistic: all our phenomena proceed in the space-time; their dynamics is projected on this screen-space which constitutes our space of reference and the privileged domain of our observations.

If we think in deterministic terms, the classification of bifurcations is an hopeless dream. Thus other research proceed toward a statistical evaluation of local bifurcation phenomena. The success of this proceeding lies upon the equality of interest that the different variables and parameters possess. We know that equality is a catch outside the abstract world, and is tied to the imperfections of our models. The statistical use of deterministic results can lead to some remarkable successes: i.e. a chaotic situation opens up on a mean regime described by the traditionnal heat equation. This kind of result can but comfort the mathematician on the valuable grounds of its work, and to encourage him to pursue research in the deterministic and non-deterministic directions, the latter leaning on the former.

The present situation of bifurcation theory is thus disarming, and one would be tempted to the helve after the hatchet. Let us chukle those who would take this infantile attitude. The era of bifurcation lies before us, and it is not forbidden to hope that the new tools we dispose of nowadays, non-standard-analysis and the technical abilities of computers, will allow some substantial progress in our manner to approach the problems and to conceive the classification. Just as the physicists at the end of the last century believe to have reached the limits of the knowledge of the physical world by men, as Maïmonide, in the 12th century, wrote in the **Guide of those who have gone astray**: " we can only give explanations of some terrestrial phenomena, but on the other hand, the constitution of celestial bodies is totally unknown to us. God has kept heaven for himself, he gave up earth to men." History has never ceased to prove prophets announcing the end of the progress od Science wrong.

Bifurcation theory is a theory of changes into the behaviour of solutions of our mathematical models. Everybody is conscious of their possible imperfections, but in general we ignore the importance of the distorsion they introduce with respect to reality. We know perfectly well that some of them have no true physical counterpart: the eternal and perfect periodic movement does not exist, and we can ask ourselves whether the horse-shoe may really appear in the midst of the physical universe. Does the study of the unfolding of homogeneous jets of high order have a meaning outside any consideration of the difficulties linked to the study of these jets and their singularities ? Can we believe in the contingency of the world, i.e. in the possibility of the physical realization of the constructs elaborated by the thought ? The response might be positive if we agree with Pascal ("the imagination will get tired of conceiving before Nature of producing") and with Shakespeare ("There are more things in heaven and earth, Horatio, that are dreamt in your philosophy").

This philosophy however becomes more pertinent as far as our knowledge of Nature increases, and as our familiarity with great numbers improves. The diversity which reigns in the animal and vegetal kingdoms, which is around a few hundred thousand different species, does not frighten us. Even inside the neuronal world which reaches the 10^{11} cells in man, we are accustomed to considering a limited number of types of morphologies, upon which Nature seems to have embroidered according to the taste of its humour. Among these morphologies, those of foliar type, which embrace the various forms of flow and of evolution by osmotic effect, seem to play a preponderant rôle. If we interpret all these morphologies as a result of bifurcation processes, it would seem that most of the bifurcations would not be practically pertinent because they would be too much degenerated. But one must not forget the character often metaphoric of the mathematical statements, and the esthetic and symbolic part that involves any thought activity: it testifies to the vigour and to the apparent liberty of the human mind to undertake and to win in the struggle for the unknown. In this sense, the assault given by the mathematicians to the universe of bifurcation will not be in vain.

It is necessary to insist on the fact that the mathematical objects which belong to the dynamical systems theory are models of the physical world, of which a lot of aspects are ignored. In particular, when bifurcation happens, degenerative phenomena occur from the mathematical point of view as from the physical point of view. It is far from being proven that in the immediate neighbourhood of singularities of bifurcation of the model under study, that would be precisely that model which correctly describes true evolutive behaviour. Indeed the energetic properties are there badly analysed, change of form if not of structure is accompagnied by a call of energy which, in some cases is of endogeneous nature when the object, destructuring itself, makes a part of its internal energy available, which he will use again to be transformed and to appear under a new morphology. There is a whole field of study to explore, whose difficulty of approach cannot be ignored.

The study of these transition phenomena intrude into two major difficulties: the multiplicity of the sub-objects which occur during destructuration, and the extreme velocity of the processes of destructuration and restructuration. This velocity masks the real continuity of the transformation. Both Aristotle and Darwin - who cited the very famous Natura non facit saltus, the philosopher Leibniz as the mathematician Poncelet a century later, all strongly assert the continuous character of evolution. In Leibniz ' **Animadversiones in partem generalem Principiorum Cartesianorum**, one finds the statement of what he calls the law of continuity, as well as a nice geometrical illustration of this law. Poncelet ignoring this Leibniz 'writing, has also asserted the power of the principe de continuité, and devoted his life as a mathematician to illustrating it in geometry. As Poncelet's work has played a major rôle in the formation of the mathematicians during the 18th century, it is not surprising that Poincaré used this famous principle in a way as clever as it was constant. The study of bifurcation is a fine example of it, but this example disturbs us. It raises the crucial problem to which no answer can as yet be satisfactorily given, of the hidden links between bifurcation and continuity.

Though in effect a parameter changes continuously, modification in morphology occurs so that, should other bifurcations occur, the initial and final morphologies seem, at first glance, completely distinct. Let us distinguish bifurcations in geometry from those in topology. The first has been known for a long time: Leibniz was the first to realize the existence of a hidden problem when he gives the progressive transformation of an ellipse into a parabola as an example of his law of continuity. Obviously there is a question here of geometrical bifurcation we can define in terms of singularities. Let us take the more elementary example of squares and of rectangles. With respect to the metric criterion, they are all different figures which can be represented by the positive quadrant of R^2, putting the length of the horizontal side in abscissa, the length of the vertical side in ordinate. In this quadrant, the 45°inclined half-line represents the squares: the area of the domain occupied by this half-line is null, thus the square is indeed a singular figure among the rectangles. Nevertheless, by déformation insensible, to use a natural expression of Poncelet again, one can pass from a square to rectangle, either expanded vertically or horizontally.

The bifurcation which is considered in qualitative dynamics is first of all topological: at first it conserves the connexity of forms. It possesses moreover the property to being able to compress until they are rendered of null measure connected domains whose boundary is made of leaves of trajectories. As by magic, bifurcation is able to unfold the singularity which has just been created by compression, and to recreate the initial form. For given analytical data, which fix the form of the greatest unfolding compatible with these data, the study of bifurcation comes to determine the different possible steps of compression or of unfolding, and the morphologies as well, which appears between the various extreme states of compression and of unfolding. In sum, a dynamical system behaves like a memory alloy: some singular values of the parameters determine the changes of form of the alloy.

When the singularities are multiple, the continuous variation is, in an initial phase, accompanied with desaggregation or separation (from a torus of multiplicity k, k distinct tori emerge) or, on the contrary, with aggregation and fusion. Of course, after separation, the set formed by the k tori is disconnected. But it is immersed in a connected phase space. The discontinuity exists only according to the hasty look, unsensitive to the realities of phase transitions, which does not take all the aspects of the dynamics into account, or which juxtaposes different morphologies, and which consequently, has suppressed some connectedness. These morphologies may sometimes be tied together through various processes, so that a partial connexity runs over the new system: the novelty is essentially here of exogeneous nature, in opposition to the novelty we have named endogeneous, resulting only from the fact of bifurcation, even if it is the consequence of the variation of parameters considered as exterior to the system.

The bifurcation is connected to a process of generation of distinct forms inside a continuous universe. That it would have been almost necessary to wait until this century to really become conscious of the phenomenon and of its importance is an index of the subtility and of the complexity of the problem. If it has only just received a precise formulation, we could not of course imagine that the scientists of the past never had some intuition of the problem. When Empedocles of Agrigente declares, in the fourth century B.C.:

At a given moment, the Whole came from the Multiple
at an other moment, it divides itself
and from the Whole came the Multiple

does not this express the most profound aspect of bifurcation ?

References

[1] V.Arnold: Chapitres supplémentaires de la théorie des équations différentielles ordinaires, Editions Mir, Moscou, 1980.
[2] C.P.Bruter: Les Architectures du Feu: Considérations sur les modèles, Flammarion, Paris, 1982.
[3] H.Poincaré: Oeuvres (11 vol.), Gauthier-Villars, Paris, 1915-1956.