BOURBAKI

In July 1935, in the village named Besse-en-Chandesse, a small set of French first rank mathematicians, joined together in a "Congress", was named by themselves Bourbaki. They formed the first Bourbaki set. Bourbaki came rapidly famous, and this is not a bourbakist joke. Nowadays, several mathematicians and historians have left evidences and studies on the group, mainly concerning his evolving structure, some of his conceptions, his future. Many people now have written on Bourbaki ; Jean Dieudonné at first, Liliane Beaulieu, are the authors among the most involved in these studies. One can easily find them on the net. Since no one deals with pure mathematics, i.e. mathematical facts and proofs, they can be easily read. After recalling some basic and well-known facts about Bourbaki, I shall try shortly to bring a very few complementary points of view to the previous studies from the semiotic point of view. Any non mathematician can read various Bourbaki' pages without difficulty and with some pleasure, as for instance the introduction to the volume titled "theory of sets" and plenty of historical notes.

The story begins in 1934 when Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, René de Possel, André Weil met in the basement of the café Capoulade in the Quartier Latin. With the adjunction of Jean Coulomb for a while, of Charles Ereshmann and Szolem Mandelbrojt, the father of the well-known fractalist, the Capoulade set became the previous Bourbaki set.

I shall first quote here one of the most important elements of this set. Note that in this paragraph, the terms set and elements do have an enriched connotation compared with the usual bourbakist one : for a bourbakist and a more generally a mathematician, sets and elements are abstract or ideal objects. Would you dare say that a man is abstract or ideal ? The quotation comes from Jean Dieudonné's address, given in Orsay on June 17 1975 to honor Henri Cartan's retirement (Jean Dieudonné : July 1, 1906 - November 29, 1992 ; Henri Cartan : July 8, 1904 - …) :

"L'année 1935 est pour nous mémorable, puisqu'à quelques semaines de distance nous fondons l'un et l'autre un foyer, et que la même année a lieu le premier Congrès Bourbaki. Je crois que l'on peut aujourd'hui *révéler* [italics are mine] que c'est toi qui a été le catalyseur des évènements d'où est sorti Bourbaki. Jeune professeur à Strasbourg, tu te préoccupais de bâtir un cours de Calcul différentiel sans faille, et avec la conscience proverbiale des Cartan, tu revenais sans cesse sur les démonstrations des théorèmes classiques, et notamment de la formule de Stokes. Tu avais retrouvé à Strasbourg André Weil et tu le harcelais de questions sur la meilleure manière d'exposer les fondements de l'Analyse ; jusqu'au jour où Weil décida qu'il fallait en finir, et que le seul moyen de te contenter était de rédiger un Traité de Calcul différentiel et intégral, qui remplacerait les ouvrages classiques et te donnerait satisfaction. On connaît la suite et ce qu'il advint rapidement de ces illusions juvéniles [Bourbaki did'nt yet prove the Stokes'formula, though it is an important tool in mathematics and physics !]. Ce que savent seuls les collaborateurs de Nicolas Bourbaki, c'est l'importance du rôle que tu as joué dans nos travaux. Sans doute, il ne fallait pas te demander trop souvent des rédactions détaillées ; mais ton esprit méticuleux ne laissait jamais rien passer lorsqu'étaient épluchés les états successifs des chapitres en préparation ; et combien de fois n'avons-nous pas été surpris par ton refus apparent de comprendre ce qui nous paraissait clair, après quoi on s'apercevait qu'en fait tu avais compris mieux que tous les autres, comme dans la fameuse séance de 1937 où furent inventés filtres et ultrafiltres. »

The disclosure done by Dieudonné was confirmed by Cartan in 1982 and, in 1991, by André Weil (May 6, 1906 – August 6, 1998) in 1991. An other motivation for undertaking the work was the fact that, in that time and in great part because of the first world war, the French mathematics was far behind the German one in particular. A verse of an unpublished poem[1], suggesting that Bourbaki would be Corsican, expresses the kind of confusion of the young team of active mathematicians. When he was young, Bourbaki wrote a few canularesque poems : one understands that jokes and leg-pulls allowed these hard working mathematicians to relax from time to time. A pleasant biography of Bourbaki was written by André Weil from whom one learns that Bourbaki was in fact from Greek ascendance. One will consult the texts of the historians in order to get more serious details about this intriguing and polycephale Nicolas Bourbaki. The canularesque aspect of the existence of Bourbaki equilibrates the deep seriousness of his work. Without any doubt, it has contributed to his fame. Without any doubt either, it bears the mark of the personality of André Weil. To cock a snook to the society is not only a work of derision, it is also the expression of a definite sense of humour.

His first scope was to write a rigorous treatise on Analysis that could be useful to everybody dealing with mathematics. In the sequel, I shall quote two papers by Dieudonné : the first one, titled "The work of Bourbaki" was published in 1970 in the American Mathematical Monthly ; the second one, "The work of Bourbaki during the last thirty days", appeared in 1982 in the Notices of the American mathematical Society. In the 1982 paper, one of the best paper written about Bourbaki, he insists repeatedly, and these are his last

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[1] Here is the complete poem : Là Pauline erre
Sur champ où naguère
Dans la langue du pays
Avant qu'il ne faillit
Nicolas écrivit
Je suis dans la bourbe, aqui.

words : "the fundamental purpose : to provide tools for the working mathematicians."

The meaning of tool is not defined precisely, and there are no explicit examples of the use of these tools in order to study the properties of some mathematical object.

Dieudonné writes in his first page :

"The conception of Bourbaki was more ambitious : starting from scratch, to lay the groundwork for <u>all</u> [underlined by Dieudonné] theories *then* [italics are mine] in existence in pure mathematics. Applied mathematics were never considered, due primarily to the lack of competence and lack of interest of the collaborators ; for some time they toyed with the idea of including probability theory and numerical analysis, but this also was soon dropped."

To reach his scope, Bourbaki had first to deal with some basis of mathematics. "Starting from scratch was meant literally, not merely, as in all treatises or monographies then in existence (even van der Waerden 's "Moderne Algebra") starting from some "naive" theory of sets. It meant including the fundamental rules of logic, whilst keeping as close as possible to the actual practice of mathematicians."

Rapidly, one thing leading up to another, optimistic, he thought at a complete mathematical treatise he called *Eléments de Mathématique*. Even if the published work contains only a part of the mathematics, and so does not handle with various very important chapters, it remains as an immense work, of great richness and value, of great beauty. The books were written in a very clear and elegant French.

The Bourbaki's treatise is divided in 10 books, each one has been divided into chapters which have first been published separately. There has been several editions and errata. The treatise does not intend at all to be a research book with almost only new results. But anyway and naturally it wants to be an up-to-date and the most well-written treatise. It can but take into account the mathematical interests of its writers. From the outside and globally, the already published "Elements" concern two main themes. The first deals with analysis and geometry under the name of Topology and under the main influence of Henri Cartan and André Weil ; the first published books are devoted to the theory of sets, the general topology, the function of a real variable, the basic data of algebra and the topological vector spaces, the integration (last chapter published in 1969, measure theory is the most present theme in this book). The books connected with the second theme appeared later ; though one of them is devoted to the study of spectral theory, they mainly concern groups and in particular Lie groups and commutative algebra (chapters 1 to 4 appeared in 1968-1969, after the publication of chapters 5 to 7 in 1964-1965). On the works of this second period, one feels the influence not only of the founders among whom Claude Chevalley (February 11, 1909 – June 24, 1984) but also of the new members of the “tribe” as they call them. Pierre Samuel was indeed one of the best mathematician at his time involved in commutative algebra.

The fact that the young future bourbakists discovered algebra through the German works played an important role on their mathematical project. Following Felix Klein, they rather thought at geometry as an application of the algebraic results. Some of Dieudonné’ mathematical works illustrate that point of view. That explains why Bourbaki did not publish any book on geometry, whichever be its qualification (algebraic, Euclidean, differential, …). During some time, Bourbaki put the algebraic point of view so forth that Choquet wrote me that “Bourbaki is an algebrist”. But Dieudonné objected to such an affirmation. In fact, most of the progresses of the geometrical knowledge may use algebra as a tool of proof, not at all as a tool of discovery.

The cover of the first publication bears the date of “1939”, but my copy gives the dates of printing : 1-2-1940 for the text, 5-2-1940 for the cover. On this cover are printed several titles ; from top to bottom one reads : “Éléments de Mathématique par N. Bourbaki, I, Première partie Les structures fondamentales de l’analyse, Livre I Théorie des ensembles [red capitals] (Fascicule de résultats)”. The drafting of the treatise has been a long and “painful” collective work. But, at least during the first years, Cartan and Dieudonné had a central position. On Cartan I shall quote Dieudonné (his address to Cartan)

“C’est à la rentrée d’octobre 2004 que nous nous sommes rencontrés, alors que j’entrais à l’Ecole, où tu commençais ton année de carré. Je crois bien me souvenir que c’est notre amour commun de la musique qui nous a rapprochés, comme il allait demeurer une composante constante de notre vieille amitié. Combien de morceaux à 4 mains n’avons-nous pas déchiffrés sur le piano de l’Ecole ; à combien de concerts ou d’opéras n’avons-nous pas assisté ensemble, rentrant souvent à pied à l’Ecole en commentant avec passion ce que nous venions d’entendre ! En contact étroit avec tout ce que la vie musicale de ce temps offrait de neuf et d’enthousiasmant, tu me faisais pénétrer dans cet univers enchanté où j’allais de révélation en révélation. Mais bien entendu les mathématiques n’occupaient pas une moindre partie de notre vie, et c’est ensemble également que nous allions assister aux cours de Picard, de Lebesgue, de ton père, ou au fameux Séminaire Hadamard ; et là aussi, ta précoce maturité et cette remarquable faculté que tu as toujours eue de pénétrer au fond des choses sans jamais te contenter des apparences extérieures, m’ouvraient sans cesse de remarquables horizons. Si bien que je puis dire sans exagération que ces échanges constants, les plus enrichissants de ma vie à l’Ecole, ont fait de moi le premier en date de ta nombreuse cohorte d’élèves ; et je rends grâce au sort qui m’a permis d’avoir sous les yeux, pendant toute ma vie d’homme, un modèle sur qui régler mes actions et mesurer le résultat de mes efforts. »

On Dieudonné I shall quote Cartan (from his paper « Le plaisir des mathématiques » written in 2003) :

« Je me sens chez moi en géométrie, plus exactement en topologie ; mais même si, aidé par Serre, j'ai particulièrement apprécié de pouvoir établir quelques connections entre la topologie algébrique et les fonctions analytiques, il n'y avait pas de partie des mathématiques qui me plaisait tellement plus que les autres. Vouloir les découper en morceaux séparés par des cloisons étanches ne peut conduire qu'à la stérilité. Naturellement j'ai souvent écrit sur des sujets précis ; mais la rédaction de Bourbaki, pour laquelle j'ai été entraîné par mes camarades (surtout par André Weil) et à laquelle j'ai donné beaucoup de temps à partir de janvier 1935, voulait autant que possible ne négliger aucune des grandes branches des mathématiques de l'époque, en les stimulant souvent. Ces longues réunions et discussions me passionnaient toujours : nous avons beaucoup travaillé ! J'ai apporté contributions orales et écrites, mais c'était toujours Dieudonné qui donnait la rédaction finale. » [Pierre Cartier played this role of writer when Dieudonné retired from Bourbaki ; the age of retirement is fixed at 50, but …];

Of course, the Bourbaki' collaborators have an excellent knowledge of most parts of mathematics and are first class technicians. But they are not only technicians. They may think at their activities, their organization, their roots, their finalities. The treatise reflects these attitudes of the mind, this power of insight. Bourbaki recognizes the influences of the previous masters ; Dieudonné writes in the 1982 paper :

"The mathematicians who had the deepest influence on Bourbaki were probably in Germany Dedekind, Hilbert and the school of algebra and number theory of the 1920's, and in France H. Poincaré and Elie Cartan. Although these great mathematicians are very different from one another, both in their style and in their fields of research, they have a common philosophy of mathematics, namely to try to solve classical problems by methods involving "abstract" new concepts, and that is also, in my opinion, the central idea of Bourbaki.

…

The influence of the German school of algebra has been underlined by Dieudonné. In the 1970 paper for instance, he says that :

"the Bourbaki treatise was modelled in the beginning on the excellent algebra treatise of Van der Waerden [*Moderne Algebra*, already mentioned, the first edition appeared in 1930]. I have no wish to detract from his merit, but as you know, he himself says in his preface that really his treatise had several authors, including Emmy Neother and A. Artin, so that it was a bit of an early Bourbaki.

Though only Van der Waerden's book is mentioned, there is a lot to bet that the Hilbert-Bernays 's huge book, *Grundlagen des Mathematik*, had a great influence on the work of Bourbaki (Vol. 1 1934, vol. 2 1939).

Going back to the 1982 paper, Dieudonné gives some Bourbaki' choices :

Summing up, we see that, in spite of its initial aim at universality, the scope of the Bourbaki treatise has finally been greatly reduced (although to a still respectable size) by successive elimination of :

1) the end product of theories, which do not constitute new tools ;
2) the unmotivated abstract developments scorned by the great mathematicians ;
3) a third restriction comes from the fact that some very active and very important theories (in the opinion of great mathematicians) still seem very far from a clear description in terms of interplay of perspicuous structures ; examples are finite groups or the analytic theory of numbers ;
4) finally, there are parts of mathematics where the underlying structures are well in evidence, but in such an ebullient state, with an unending influx of powerful new ideas and methods that any attempt at organization is doomed to almost immediate obsolescence : think of algebraic and differential topology, or algebraic geometry, or dynamical systems."

Note that, besides of being quite aware of his limitations, being not aware of some weaknesses, Bourbaki did not pretend to be infallible. He did what he considered the best given the scope he had in mind.

Facing a great diversity of events and facts, one of the first steps of the thought is to undertake some classification. In the past centuries, the mathematical corpus was divided into great chapters called arithmetic or number theory, geometry, algebra, analysis. The British school of philosophers have almost always been interested in the study of the ways the mind operates. This state of mind had a great influence on the work of some mathematicians : like Frend, Playfair or Peacok for instance, mainly during the first half of the 18th century, they begin to look at some intrinsic properties of the proceedings of thought that appear in mathematics. This marching opened on the creation of logic on one hand, on putting in the lime-light the axiomatic rules of algebra on the other hand. The German mathematical school in particular largely developed algebra in the second half of the 18th century. In algebra, axiomatic presentations used to be given. In 1899, in his *Grundlagen der Geometrie*, Hilbert extended the axiomatic approach to geometry. His work influenced all the mathematicians. In 1910, the American geometers Veblen and Young published their first volume of *Projective geometry* (Oscar Veblen was professor in Princeton, and considered by Norbert Wiener as 'one of the fathers of American mathematics"). Apparently, Bourbaki had no knowledge of that book. It is worth reading its two first pages, where we see how the proceeding followed by Van der Waerden and Bourbaki was natural. Bourbaki's treatise is the unfolding of Veblen-Young's point of view. The reader might forgive such a long quotation from their pages, it begins with the line 9 of the book – italics are the authors ‘ones :

1. **Undefined elements and unproved propositions**

…

‘Some of the elements and relations, by virtue of their greater simplicity, are chosen as fundamental, and all other elements and relations are defined in terms of them. Since any defined element or relation must be defined in terms of other elements and relations, it is necessary that one or more of the element and one or more of the relations between them remain entirely *undefined ;* otherwise a vicious circle is unavoidable. Likewise, certain of the propositions are regarded as fundamental, in the sense that all other propositions are derivable, as logical consequences, from these fundamental ones. But here again it is a logical necessity that one or more of the propositions remain entirely *unproved ;* otherwise a vicious circle is again inevitable.

The starting point of any logical treatment of geometry (and indeed of any branch of mathematics) must then be a set of undefined elements and relations, and a set of unproved propositions involving them ; and from these all other propositions (theorems) are to be derived by the methods of formal logic. Moreover, since we assumed the point of view of formal (i.e. symbolic) logic, the undefined elements are to be regarded as mere symbols devoid of content, except as implied by the fundamental propositions. Since it I manifestly absurd to speak of a proposition involving these symbols as self-evident, the unproved propositions referred to above must be regarded as mere *assumptions*. It is customary to refer to these fundamental propositions as axioms or postulates, but we prefer to retain the term *assumptions* as more expressive of their real logical character.

We understand the term *a mathematical science* to mean *any set of propositions arranged according to a sequence of logical deduction.* From the point of view developed above such a science is purely *abstract*. If any concrete system of things may be regarded as satisfying the fundamental assumptions, this system is a *concrete application* or *representation* of the abstract science. The practical importance or triviality of such a science depends simply on the importance or triviality of its possible applications. These ideas will be illustrated and further discussed in the next section, where it will appear that an abstract treatment has many advantages quite apart from that of logical rigor.

> 2. **Consistency, categoricalness, independence. Example of a mathematical science.** The notion of a *class** of objects is fundamental in logic and therefore in any mathematical science. …”

The authors add in the note : “* Synonyms for *class* are *set, aggregate, assemblage, totality ;* in German, *Menge;* in French, *ensemble.*”

We may call Veblen and Young a 2-Bourbaki set, Artin-Noether-Van der Waerden (and a few other eventually) a 3-Bourbaki set, etc. The original

Bourbaki set has founded his treatise on the more or less naïve Zermelo-Frankel theory of sets, has tried to go down to a pure logic foundation, and had to give up along this direction.

There is one point on which I disagree with the Veblen-Young point of view : the use of the term “logical”. It is a superficial terminology which comes from the fact that it expresses and shortens a non evident internal bio-physical process which allows to pass from a proposition to an other. In that process, the presence of causality is the main feature ; that is why I have always replaced the term “logical” by the one of “causal”. On the other hand, it is quite significant that little by little the qualification of “logical” given to mathematics has been given up. Unconsciously, the reality comes up more and more present into the language which strives for describing the environment and its evolution. Of course, most mathematicians were aware of the misunderstanding of mathematics. In his marvellous well written book, *Hommes, formes et le nombre*, Arnaud Denjoy (January 5, 1884 – January 21, 1974) noticed :

« Toute connaissance, avant d’être irrévocablement acquise à l’esprit, doit s’élever jusqu’au dernier degré de ces trois degrés successifs : l’observation, l’expérimentation, la déduction. Pour les gens trop sommairement informés, toute l’évolution des mathématiques se déroule exclusivement sur ce dernier degré. »

Now comes a question. Does the painter Georges Braque (May 13, 1882 – August 31, 1963) have ever read Veblen-Young’ book, or rather Hilbert’s one ? That could be possible, given the Braque’s interest in geometry. But it is surely without reference to the mathematics which were practiced in his time that the linguist Louis Hjelmslev wrote in 1929 :

“To describe the language as being essentially an autonomous entity of internal dependencies, or in one word, a structure”.

An other use of the word “structure” in linguistics among the first was due to Roman Jacobson in a 1932 article : “Zur Strucktur des russischen Verbums”. Note that Jacobson considers C.S. Peirce has the founder of the structural linguistics. In his foreword to the French book *Essai de linguistique générale*, the chapters in the book are translations of some Roman Jacobson’ best articles, Nicolas Ruwet writes that Jacobson, “about his orientation towards the structuralism, [underlined] the crucial role played by creators like Picasso, Joyce, Stravinski or Braque (whose he quotes the sentence : ”I do not play reliance on things, but on relations between things”) (1962, Selected writings, p.632).”

Indeed, the structure describes the relations between the elements of a construction. The use of the term does fit the way mathematical objects are build : the set of axioms which defines the connections in the elements of a group for instance sums up its structure. To speak of a mathematical structure is quite pertinent.

In his *Souvenirs d'apprentissage* (p. 120), André Weil wrote :

"Dans l'établissement des tâches que Bourbaki allait entreprendre, un progrès notable fut accompli par l'adoption de la notion de structure, et de la notion d'isomorphisme qui lui est liée. Rétrospectivement , celles-ci semblent banales, et d'un contenu mathématique assez mince tant qu'on n'y ajoute pas les notions de morphisme et de catégorie. A l'époque de nos travaux c'était une lumière nouvelle jetée sur des sujets où régnait encore une grande confusion. ; le sens même du mot « isomorphisme » variait d'une théorie à l'autre. Qu'il y eût des structures simples de groupes, d'espace topologique, etc., puis des structures plus complexes, depuis les anneaux jusqu'au corps des nombres réels et aux espaces vectoriels topologiques, cela n'avait pas été dit avant Bourbaki, que je sache, et il fallait le dire. Quant au choix du mot structure, mes souvenirs sont en défaut ; mais à cette époque, il était déjà entré, je crois, dans le vocabulaire des linguistes, et je conservais des contacts avec ce milieu, et tout particulièrement avec Emile Benveniste ; sans doute n'y avait-il pas là qu'une simple coïncidence. »

In the middle of the thirties, the term structure seems to be have been used without provoking restless interrogations. The fact that André Weil does not remind when the term entered the Bourbaki's vocabulary is quite significant. The word seems to appear for the first time in 1935, only in the title of a paper by Garrett Birkhoff, "On the structure of abstract algebras" - it appeared in the Proceedings of the Cambridge Philosophical Society. The same year, referring to the previous article, Oysten Ore, professor at Yale, published an article in the Annals of Mathematics titled 'On the foundations of abstract algebra" where he gives the premises of what the mathematicians will intend by mathematical structure. Here are his first lines : :

'It is obvious to any connoisseur of abstract algebra that by the study of the structure of the principal domains of algebra like group theory, ideal theory, hypercomplex systems, rings, moduli, etc. one arrives at a great number of results showing close relationship and similarities.'

Being in Paris for a while, he published a book, *L'Algèbre Abstraite*, in 1936. He writes – italics are Ore's :

« il faut observer qu'il existe aussi des questions fondamentales communes à tous les types. Pour chaque système algébrique nous cherchons des *théorèmes de décomposition ou de structure* exprimant des relations entre un système et ses sous-systèmes spéciaux. »

In 1938, Valère Glivenko, professor in Moscow, published a book, *Théorie générale des structures* :

«Ce fascicule contient l'exposé d'une branche moderne de l'Analyse générale. L'intérêt des études qui s'y rattachent consiste en ce qu'elles facilitent l'explicitation des fondements e plusieurs disciplines mathématiques d'une façon permettant de comprendre ce qui est commun à ces disciplines et ce qui

leur est spécifique. Il s'y agit des fondements de l'Algèbre abstraite, de la géométrie projective, de la théorie de la mesure, de la théorie des probabilités, et d'autres. »

The theory rather refers to lattices. At the same time appears the thesis of Albert Lautman *Essai sur les notions de structure et d'existence en mathématiques* and a complementary study, *Nouvelles recherches sur la structure dialectique des mathématiques*. Hermann was the editor of all these books, the *Eléments de mathématiques* as well.

So, in this period, the term "structure" was not only getting commonly used in linguistics, mathematics, but also in social science. In the chapter VII of his *Essays in Linguistics*, titled "Structure and function in language", J.H. Greenberg quotes A.R. Radcliffe-Brown who, in 1935, in his article "On the Concept of Function in Social Science" wrote :

"The concept of function as here defined thus involves the notion of structure consisting of a set of relations amongst unit entities, the continuity of the structure being maintained by a life process made up of the activities of the constituent units."

Claude Levy-Strauss has particularly illustrated the use of the term structure in social science with his book *Les structures de la parenté* (1948). It is sure that the André Weil 's help contributed to the fame of the book.

The book whose editor was François Le Lionnais had a great fame. The first edition of his book *Les Grands courants de la pensée mathématique* was in 1948 ; the second world war was just behind. It is a huge book where most of the best French mathematicians, physicists and philosophers in science at that time wrote an article : Emile Borel, Georges Bouligand, Bourbaki, Robert Deltheil, Jean Dieudonné, Arnaud Denjoy, Paul Dubreil, Elie Cartan, Maurice Fréchet, Paul Germain, Roger Godement, André Lentin, Maurice Janet, Paul Montel, Robert Fortet, André Sainte-Lagüe, Andréas Speiser, Georges Valiron, André Weil, have now their name written in mathematical history. In that book, Bourbaki was present four times through the writings of himself, Dieudonné, Godement and Weil, even more than four times since Dubreil and Germain have been temporary his collaborators.

It seems to me that Bourbaki's article has been mainly written by Henri Cartan and Dieudonné. For the first time maybe, the grandiose term "architecture" was used : *L'architecture des mathématiques* was indeed the title of the article. Bourbaki was proud to have been able to describe the mathematical world in both a synthetic and analytic way ; so he thought at that time. Getting older, his first "illusions", the André Weil 's word, vanished. The fact is that Bourbaki has been fascinated by the term structure, and indeed, one of his first work was to give a precise definition of a mathematical structure. That work was only the continuation, the unfolding of the work undertaken by his predecessors and contemporaries. But it has been a long and "painful" collective work, and we can understand that these yet young mathematicians were proud of the result. Bourbaki and his collaborator Pierre Samuel use to distinguish three categories of mathematicians : the great discoverers, the sound-bodies, the piece-workers. Bourbaki, the strong personality of his speaking-trumpet Dieudonné (Denjoy wrote :

« Frappés de l'aisance avec laquelle M. Dieudonné cite de mémoire Nicolas Bourbaki, et afin de fixer cette polyvalence dans l'une de ses déterminations possibles, nous trouverons commode de voir Bourbaki sous les traits de M. Dieudonné. »)

have been a sound-body of the term structure. The outstanding quality of his collaborators, the character secret and canularesque of his person, the high value of his writings, their abstractness and their difficult reading by the non-professional adding to his mystery, have made of Bourbaki a particular mediatic entity. From his mouth, the term "structure" gained a large audience, only for a time, being the character natural of the procedure and the limitations of its use. There are fashionable words : structure has been one of these, as fractals is today. It would be interesting to have a mathematical study of the audience of such terms. I shall conclude about the history of "structure", again by a quotation from Dieudonné, in his foreword to a new edition of Albert Lautman's works titled *Essais sur l'unité des mathématiques* and published in 1977 :

« Le mot "structure" est un de ceux qui ont été le plus galvaudés au cours des dernières décennies ; mais pour les mathématiciens il a acquis un sens parfaitement précis. En 1935 ce sens n'avait pas été encore complètement explicité ; mais la réalité qu'il recouvre était tout à fait consciente chez de nombreux mathématiciens, notamment chez tous ceux qui s'inspiraient des idées de Hilbert sur la conception axiomatique des mathématiques. Le point essentiel, dans cette conception, est qu'une théorie mathématique s'occupe avant tout des *relations* entre les objets qu'elle considère [cf Veblen-Young 'quotation !], bien plus que la nature de ces objets : par exemple, en théorie des groupes, il est le plus souvent secondaire de savoir que les éléments du groupe sont des nombres, des fonctions ou des points d'un espace ; ce qui importe c'est de savoir si le groupe est commutatif, ou fini, ou simple, etc. Ce point de vue a tellement imprégné les mathématiques depuis 1940 qu'il est devenu assez banal ; mais ce n'était pas encore le cas au moment où écrivait Lautman, et il y insiste à plusieurs reprises, comme par exemple lorsqu'il souligne l'identité fondamentale de structure entre l'espace hilbertien, composé de fonctions, et l'espace euclidien usuel. Plus remarquable encore est le long passage qu'il consacre à ce que l'on appelle maintenant la notion de revêtement universel d'une variété (on disait à l'époque « variété de recouvrement universelle»). La « montée vers l'absolu » qu'il y discerne, et où il voit une tendance générale, a pris en effet, grâce au langage des catégories, une forme applicable à toues les mathématiques : c'est la notion de « foncteur représentable » qui joue aujourd'hui un rôle considérable, tant dans la découverte que dans la structuration d'une théorie. »

One are indebted to Bourbaki for having precised the notion of algebraic structure and introduced the one of topological structure (owing to Cartan's filters). The fact is, however, that he has not completely done the job. He did not pay a great attention to geometry, leaving apart the Whitney's axiomatic for linear independence which appeared in 1935. He never put in evidence that the axiomatic of vector spaces is a formal translation of the basic properties of Euclidean geometry, a rather unknown fact that is important from the pedagogical point of view. These small deficiencies come from the fact that his first interests were in analysis.

In any science, the work of clarification and foundation is from time to time quite necessary. Given the imposing unfolding the mathematical universe since Bourbaki, the setting up of new local structures seems to be necessary. As a coincidence, I shall quote here the titles of two books which appeared very recently (2007) : *Projective Group as Absolute Galois Structures with Block Approximation* and *The Structure of the Rational Concordance Group of Knots.* An other significant example is the announcement of a workshop under the guidance of some best mathematicians in order to "understand the structure of relative Symplectic Field Theory", in order to "work towards building rigorous foundations of the theory."

Bourbaki, and Dieudonné, have been a lot criticised. I shall not enter this subject here ; my book *De l'intuition à la controverse* (1987), reviewed without any remark by Dieudonné, evokes some of these critics. When one reads in detail their writings, one can but recognize that Bourbaki and Dieudonné were quite conscious of most of them. I would only like take up again an observation I made in the book just quoted above :

"L'examen de la structure d'ordre interne des objets est loin d'être vain : car cette structure est intimement liée au mode de construction de l'objet, à son développement organique par croissance organisée. … Cependant, il faut reconnaître aux modèles structurels de présenter une insuffisance de taille : ce genre d'études ignore tout du moteur interne qui établit le régime de croissance des objets. »

Bourbaki has given us a kind of partial radiography of the mathematical corpus, of this organic construction. This radiography has an instantaneous character : with respect to geological times, a forty years duration is quasi an infinitesimal. But how does this mathematical corpus evolve, for what reasons, according to which processes ? The representation of the physical reality into the symbolic formalism of mathematics leads to the transposition of questions about that reality into problems to be solved. One class of studies refers to rather the static knowledge of the properties of the representations. An other class of studies refers to queries concerning the immediate or not future, and having a dynamic connotation. I have to share my goods and chattels into my three children ; how much do I have to give to each of them, in a moment or in a more distant future? To take up challenges under the form of pertinent representations of the reality on one hand, under the form of resolution of equations on the other hand, might have been until now the two most important movers of the evolution of the mathematical corpus. All these aspects have not been taken into account by the Bourbakist philosophy, and they might have an important pedagogical and educational effect.

There are several levels of abstraction in the symbolic representation. Elements of a mathematical group can be men of tribe (cf Levi-Strauss), points, numbers representing themselves geometrical movements. In the present cases, these elements as symbolic objects do have a physical connotation, a proper semantic. You can also try to think at the elements of the group and try to find properties without referring to any semantic : that was the idea of Hilbert that Bourbaki intended to follow. But that is an utopian project since the realization of such a work has to be made by some human biological minds. From this point of view, the scrutiny of the vocabulary introduced by Bourbaki is revealer. They have introduced the qualifiers "Artinian" and "Noetherian" : each one refers to a non abstract human being. They (Cartan in fact) have introduced in mathematics the term "filter": everybody knows what is a physical filter, and the mathematical definition of the term refers to a common property between the usual material object and the mathematical one. They don't have introduced the term "bixaptek" because it does not correspond until now to any reality in the languages they knew. Instead, they used the medical term "injection", from which were build "surjection" and "injection", reflecting physical transformations. Other mathematical terms that we had just met, like "hilbertien", "recouvrement" or "revêtement" refer to the physical reality given by our senses. The conception of mathematics as an abstract physics leans on the manner the mind proceeds to figurate the reality. The progresses in mathematics as in the other fields of thought arise from one part from the internal study of the field, but first from a pertinent representation of the reality giving birth to the basic axioms and facts, and it happens sometimes that a reflection of the semantics of the words commonly used in the field allow a better understanding of the concepts and an extension of the theories based on these concepts.

Very few publications today refer to Bourbaki. In that cases, they mostly refer to his last volumes of Lie groups. A typical example of this abandonment is a recent book on integration theory (by G. Bartle) based in particular on some new fittings up of the old Riemann integral : since "*no measure theory* nor virtually no *topology* is required", the reference to Bourbaki'tools in the field becomes useless. But it is interesting to underline some similarities between this book and Bourbakis's ones : as also Bourbaki-Dieudonné said, "The author makes few claims for originality", both of them refer to many collaborators.

A special point of this treatise has to be underlined. It contains one of the best survey on the history of mathematics. It has been written on the impulse of André Weil. In his *Souvenirs d'apprentissage* (p. 119), he mentionned that

"A ma grande satisfaction, (car l'histoire des mathématiques, ou pour mieux dire la lecture des grands textes mathématiques du passé, me fascinait de longue date) le principe fut adopté de faire suivre chaque chapitre, non seulement d'exercices plus ou moins difficiles, mais aussi d'un « laïus » historique, première annonce des « Notes Historiques » qui allaient contribuer à conférer son caractère distinctif à notre ouvrage."

May be, this attraction for the history might be connected to the constant and ancestral study of one the most well known historical and philosophical book, the Bible.

To conclude, I would like to quote again Henri Cartan "*Le plaisir des Mathématiques*":

"Dans le discours que j'ai prononcé le premier février 1977 à l'occasion de la réception de la Médaille d'Or du CNRS, j'ai tenté de défendre la thèse selon laquelle les mathématiques relèveraient plutôt de l'art que de la philosophie : il est vrai qu'une théorie mathématique bien faite inspire un sentiment esthétique, comme une belle construction en architecture ou en musique, et que les qualités esthétiques d'une belle théorie en facilite la diffusion, la rendant apte à une utilisation efficace. »

The Bourbaki treatise bears the imprint of his main authors, the one of their artistic and human sensitivity. The work of art of Bourbaki will remain as a an important jewel in the history of mathematics.

Short Bibliography

www.**bourbaki**.ens.fr/ ,

http://www.its.caltech.edu/~ljpk/bourbaki.html http://planetmath.org/encyclopedia/NicolasBourbaki.html#BW

www-groups.dcs.st-and.ac.uk/~history/References/**Bourbaki**.html

Additional information

1) From « Les voix de la liberté » by Michel Winock, Editions du Seuil, Paris 2001, page 501:

L'ANNÉE TERRIBLE - 1870-1871

de décembre. En janvier, les Allemands occupent la France de l'Ouest jusqu'à Alençon ; au nord, l'armée de Faidherbe, vainqueur à Bapaume, a été arrêtée vers le sud à Saint-Quentin ; à l'est, le général Bourbaki a reçu la mission de délivrer Belfort, défendu par le colonel Denfert-Rochereau, mais, si Belfort résiste, l'armée de l'Est finira par une débâcle en Suisse, après que Bourbaki aura tenté de se tuer

ON THE CLASSIFICATION OF NUMERICAL INFINITES

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Abstract: In this elementary article, we note a thoughtlessness and confusion which appear in Bourbaki's works devoted to set theory. We then introduce and specify the definitions of ordinally and cardinally equipotent. There follows, by their size, a first fairly obvious classification of infinite countable sets with a regular and uniform character.

A. A thoughtlessness and inattention on the part of young Bourbaki

1. In 1939, the young Bourbaki[1] published his first "*Fascicule de résultats"* [1] devoted to set theory. Most of this text was included in the last edition (1957) of his work "*Théorie des ensembles*" [2].

Bourbaki used the qualifier "positif" to describe certain numbers. Nowhere was the meaning of this term explained.

Bourbaki describes as "positif" an element of the ordered set of symbols {1, 2, .., n, n+1, …} also called integers. Dedekind [3] in 1887 called them "*natural numbers* or *ordinal numbers* or simply *numbers*".

Dedekind noted all of these integers, "naturals or ordinals", indifferently by the bold letter **N**. Bourbaki took up this notation.

It does not take into account the two possible meanings of the integer number noted by Dedekind. According to the first, these so-called "natural" integers must be understood as representations associated with the energetic properties of objects or events in the real world. The expressions a "thirteen hectare wheat field", a "five kilogram dumbbell", have an energetic meaning that physicists can establish. Rather than natural, it seems preferable for this physical reason to call them cardinals, which is what Bourbaki does in his 1957 book (E III.30 §4: "*un cardinal fini s'appelle aussi un entier naturel*"). As it is accepted that any presence in the real world carries energy, 0 which represents an absence of energy cannot appear in the list of symbols associated with this first set of cardinals. For the sake of clarification, we will note $\mathbf{N_e}$ this set. Its cardinal or size, which here means the quantity of its elements, noted aleph index zero ($\aleph_0$), hardly speaks to our senses.

In the second interpretation, the integers are ordinals, they characterize a position within a set, an order between objects or events existing in the real world. We could use symbols other than numbers to designate these positions. To stay in tradition, we will note $\mathbf{N_o}$ this totally ordered set of all ordinals, a set that can be

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[1] Extract from « Archives Chevalley »: « Pieusement décédé le 11 Novembre 1918 (jour anniversaire de la Victoire), en son domicile de Nancago. L'inhumation aura lieu le Samedi 23 Novembre 1968, à 15 heures au cimetière des Fonctions Aléatoires (métros Markov et Gödel). … « Car Dieu est le compactifié d'Alexandre de l'univers » *Groth.* IV.22 »

represented by a drawing, an unlimited chain. If we consider that we are classifying real, present objects, a symbol which would be associated with an absent object should not be part of the list of symbols.

In our daily reality, we take into account that objects can be a priori absent, an absence represented by the symbol 0. In this case, we will note by ℕe (respectively by ℕo) the set of cardinal integers (respectively ordinal) increased by 0.

When, for the sake of writing simplification, we do not need to distinguish between the ordinal or cardinal character of numbers, we will simply use the notations **N** or ℕ.

Let's return to Bourbaki. He writes at the end of page 39 of his book: “L’ensemble **N** des entiers positifs…”, then on page 40, “l’intervalle (0, n - 1) de l’ensemble **N**”. In other words, the new symbol 0 is here part of **N**, in contradiction with the previous definition of **N**.

This is a first “slip” of the young Bourbaki, we find it in the older Bourbaki. Unfortunately, he is no longer present to offer us reasons for this.

2. It is not in 2.9 as he indicates in the terminological index but in 2.10 that Bourbaki defines what a one-to-one application is. On page 38, he gives the definition of the term equipotent: “Deux ensembles E et F sont dits *équipotents* s’ils peuvent être mis en correspondance biunivoque”.

Further on, at the end of page 39, he speaks of “un ensemble *infini* ”. The meaning of the term infinite is unfortunately not specified here.

However, we already find it in Dedekind: “A system (= a set of elements) is said to be *infinite* when it is similar to a proper part of itself (32). » Similar corresponds to what we call injective: “to different elements *a* and *b* of the system *S*… corresponds different transforms *a’*…, *b’*”.

Bourbaki states, page 40, that “tout ensemble *infini dénombrable* est équipotent à **N**”, countable meaning “équipotent à une partie de l’ensemble **N** des entiers positifs”. Recall that equipotence implies the presence of a bijection.

3. Let $\mathcal{P}$, then be the spectrum of **N**, its subset of prime (cardinal) numbers. It is an infinite set of numbers (Euclid), more precisely an infinite digital "Gruyere" containing a countable infinity of holes, as many holes as there are prime numbers, but the size of each of them is not known to us a priori.

$\mathcal{P} = \{2, \ldots, p_i\, p_{i+1}, \ldots\}$ being an infinite countable set of cardinal numbers, by ordered construction, the existence of a bijective application of $\mathcal{P}$ in $\mathbf{N_o}$ seems assured (we make the nth of $\mathcal{P}$ correspond to the *n*th of $\mathbf{N_o}$ and vice versa), - for at the moment we can only partially construct this bijection since our current knowledge of the content of $\mathcal{P}$ is only partial. Nevertheless, equipotence *according to the order* of $\mathcal{P}$ and $\mathbf{N_o}$ is recognized.

Now consider ∑ = [2,…, n] as a segment of $\mathbf{N_e}$. *n* being greater than 3, we easily show that the restriction $\mathcal{P}$(∑) of $\mathcal{P}$ to this segment has a cardinality strictly lower than that of the segment: it follows that the cardinality of $\mathcal{P}$ is lower than that of $\mathbf{N_e}$. The application of $\mathbf{N_e}$ in $\mathcal{P}$ is a multiform surjective projection in the sense that every element *n* of $\mathbf{N_e}$ projects onto *p*(*n*) elements of $\mathcal{P}$: this cardinal number of projections is at least equal to 1. Under these conditions, we do not cannot put $\mathbf{N_e}$ and $\mathcal{P}$ in one-to-one correspondence. According to the uniform definition of equipotence given by Bourbaki, $\mathbf{N_e}$ and $\mathcal{P}$ are not equipotent.

4. *In conclusion*, if we stick to integers considered simply as elements of a countable infinite set, the assertion on page 40: " Tout ensemble *infini dénombrable* est *équipotent* à **N**" therefore does not hold. always. We are faced with a contradiction, due to confusion.

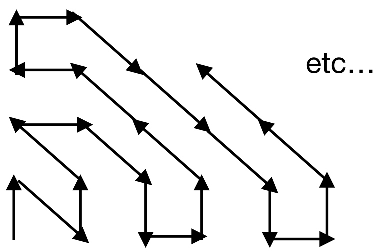
We must distinguish *equipotence according to the order* from *equipotence according to the cardinal*, one not necessarily implying the other a priori. A mass and a position are two objects of different nature. A position is an instantaneous location on a trajectory, the mobile of the same mass can possibly return to infinity through this same position. This is an inattention on the part of Bourbaki in the writing of his treatise.

We can then adopt the following definitions:

Definition 1: The orders of two ordered sets E and F are compatible for the map ß of E in F if for any pair of elements (*x*, *y*) of E where *y* is a successor of *x*, ß(*y*) is a successor to ß(*x*) in F.

If this application ß is one-to-one, E and F will be said to be *equipotent according to their orde*r or *ordinally equipotent* or more simply *ordipotent*.

Example: Consider the elements of **N** x **N** whose order is that of the vertices of a chain defined as a trajectory immersed in the real plane, passing through the points with integer coordinates of this plane - the vertices of the chain -, of so that, along a diagonal, the sum of the abscissa and the ordinate is constant. Here is the start of the visualization:



By the application ß, this chain is spread over that of **N:** we obtain equipotence according to their order between **N** x **N** and **N**. (We will compare this elementary geometric demonstration with that of Bourbaki in E.III.48).

This application ß is an application relating to order. There is another application μ relating to quantity, that of the number of elements of **N** x **N** which are projected onto a given element *n* of **N**: it is established from the number *n* of edges which join (1, *n*) to (*n*, 1): μ[(1, *n*), (*n*, 1)] = *n* - we will note the opposite orientation, according to the parity of *n*, of the chains which join these pairs of elements.

So that the length in **N** x **N** of the path which, starting from (1, 1) ends at (*n*, 1), is $n(n + 1)/2 + (n - 2)$ if *n* is odd, $n(n - 1)/2 + (n - 2)$ when n is even. The value of this length measures the weight of the crushing of **N** x **N** on **N**.

Definition 2: We will say that two sets E and F are *cardinally equipotent* or or *cardiopotent* or better *equicardinal* if there exists no part other than E itself in one-to-one correspondence with F. We will also say that they are the same size $\mathscr{T}$.

The size of a set is what Bourbaki calls his Cardinal with a capital C, the evaluation of the number of its elements. $\mathscr{T}$ only has one letter, Card four. We traditionally ask:

$$\text{Size of } \mathbf{N_e} = \mathscr{T}(\mathbf{N_c}) = \aleph_0$$

The two previous definitions clearly show the difference between **N** x **N** and **N**. We will easily evaluate the number of parts of **N** x **N** cardiopotent to N, or of the same size as **N**. By considering only the parts of the form [*n*, **N**] where *n* travels through N, we have the classic relation:

$$\mathscr{T}(\mathbf{N} \times \mathbf{N}) = [\mathscr{T}(\mathbf{N})]^2 = (\aleph_0)^2$$

We ignore here the parts constructed by joining segments of various $[a_n, b_n]$ belonging to [*n*, **N**], the indices *n* themselves traversing parts of **N**.

Definition 3: Two unordered sets E and F being given, we will call a *multiform projection* of E on F a map μ of E in F for which exists at least one element of F image of several elements of E. Such a projection defines a priori three parts of E, the one whose elements of E have no other image than the empty element of F, the one whose elements of E have a single image, the one whose elements of E have several images.

Definition 4: If, two sets E and F being given, there exists a multiform projection of E on F, E will be said to be *larger* than that of F.

B. Elementary classification of countable sets

Understanding infinity is not trivial. Cantor and Russell among others, specialists in certain series, in particular that of Riemann, know something about it. The infinite, whether physical or intellectual, is located somewhere at the bottom of a very distant fog, located within a physically open space it seems to us, and not closed. We are going to define the mathematical notion of gruyère. As trivial as it is here, a classification of infinite digital gruyeres by their properties could perhaps shed a little light on the rich diversity of this dark distance, infinity[2].

Definitions 5: We will call *digital gruyere* $\mathcal{G}$ a part of **N** formed from **N** in which we have "dug" that is to say removed numerical holes: a *digital hole* is a *connected interval* [a, b] or a segment of **N**: the elements of the interval form a totally ordered set, any element x of the interval other than a is the immediate successor of an element y_x of the interval considered - so that $x = y_x + 1$. Among these digital holes, the *singular* intervals [z] are present. We will call numerical rut in **N**, the union $\mathcal{O}$ of a set of disjoint numerical holes: a numerical gruyere is therefore the intersection of N with a numerical rut $\mathcal{O}$. **N** could also be called, according to one's taste, the *perfect* gruyere, or *pure*, or *complete*.

Consider for example the two gruyères made up, the first, $\mathbf{N_p}$, of even numbers, the second, $\mathbf{N_i}$, of odd numbers. They have the same number of holes of the same size, are cardinally equipotent, and according to the apparent order also equipotent to $\mathbf{N_o}$. The parity of their elements distinguishes them, as well as this elementary algebraic property: the operation of addition between two and any number of elements is stable for even gruyere. This same operation is not defined for odd gruyere. For this gruyère, only the addition operation between an odd number of elements is defined and stable. On the other hand, the exponentiation operation is stable in the two gruyeres.

Furthermore, by construction, $\mathbf{N_e}$ is the union, the disjoint sum of $\mathbf{N_p}$ and $\mathbf{N_i}$, each of these last two cardiopotent sets is a strict part of $\mathbf{N_e}$.

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[2]Victor Hugo in his magnificent poem entitled « Magnitudo Parvi » évoque « Le sombre écrin de l'infini », and further perhaps remembering Pascal, speaks of « l'effarement de l'infini ». Pascal writes elsewhere :« Tout ce qui est incompréhensible ne laisse pas d'être: le nombre infini, un espace infini, égal au fini. »

We must therefore put down:

$$\aleph_0 = \mathcal{T}(\mathbf{N_e}) = 2\ \mathcal{T}(\mathbf{N_p}) = 2\ \mathcal{T}(\mathbf{N_i})$$

We will also write: $\mathcal{T}(\mathbf{N_p}) = \mathcal{T}(\mathbf{N_i}) = \aleph_{(1,1)}$

The previous gruyeres are uniformly 1-hole gruyeres.

Definitions 6: More generally, a gruyere is uniformly c_u-*holey* if all the holes have the same cardinal c_u. We will denote it $\mathcal{G}(c_u)$ and we will say that it is of *type u*.

We will call c_t-separator segment any connected segment of size $c_t,\sum(c_t)$, separating two consecutive holes. A gruyere cheese is regularly c_t-holey if all its separator segments have the same cardinal c_t. We will denote it $\mathcal{G}(c_r)$ and we will say that it is of type *r*.

We will denote $\mathcal{G}(c_u\ c_r)$ for a uniformly and regularly holey gruyère and we will say that it is of the ur type.

We note:

$$\mathcal{T}(\ \mathcal{G}(c_u,\ c_r)\ ) = \aleph_{(c_u,\ c_r)}$$

Its complement in pure gruyere **N** is gruyere $\mathcal{G}(c_r,\ c_u)$ of cardinal $\aleph_{(c_r,\ c_u)}$ and we have:

$$\aleph_0 = \mathcal{T}(\ \mathcal{G}(c_u,\ c_r)\ ) + \mathcal{T}(\ \mathcal{G}(c_r,\ c_u)\ ) = \aleph_{(c_u,\ c_r)} + \aleph_{(c_r,\ c_u)}$$

We will denote by $\aleph_0\ /(c_r + c_u)$ the cardinal number of the set of connected segments of size $(c_r + c_u)$ which constitute $\mathcal{G}(c_u\ c_r)$ and $\mathcal{G}(c_r,\ c_u)$.

These simple numerical data therefore make it possible to classify these first gruyeres.

Another family of interesting gruyeres is that $\{I_p,\ p \in \mathcal{P}\}$ of prime ideals of **N**: ordipotent certainly, but since $I_p = \mathcal{G}(p\text{-}1,1)$, $(I_p \cap I_q) = I_{pq}$, their size $\mathcal{T}(I_p) = \aleph(p\ \text{-}1,\ 1)$ decreases as the value of the prime integer p and the size of the holes increases. We can note by $\partial(i+1,i) = (p_i\ \text{-}1)/p_{i+1}\ \text{-}1)$ the rate of decrease of $\mathcal{T}(I_{p+1})$ relative to $\mathcal{T}(I_p)$.

There are a thousand ways, euphemism, of imposing additional constraints to diversify these first families. The question is that of their relevance, of their interest for mathematics and physics.

Bibliography

[1] N. BOURBAKI, Théorie des ensembles (Fascicule de résultat), Hermann, Paris, 1939.
[2] N. BOURBAKI, Théorie des ensembles, Hermann, Paris, 1957.
[3] R. DEDEKIND. Essays on the theory of numbers, Dover Publications, New York, 1901.