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Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 76, No. 6, June-July, 1969

AN ALGEBRAIC STRUCTURE FOR GRAPHS ON THE FACTORIZATION OF GROUPOIDS

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I. Neutral Partial Groupoids: Representation. The linear graphs have been until now considered as combinatorial objects without well-defined bonds with the classical algebraic structures.

It would be interesting, from the conceptual point of view, to succeed in defining graphs by means of an algebraic structure, in order to throw some light on the nature of combinatorics. This is the purpose of the note.

Let us denote by $p(E, \circ) = p(B \to E)$, where $B \subseteq E \times E$, the set E with a law of composition $\circ$ defined on some couples B of its elements. $p(E, \circ)$ is called a partial groupoid or p-groupoid. If x (at the left) and y (at the right) cannot be composed, we write $x \circ y = \varnothing$. We define the valency $v(x)$ of an element x of a p-groupoid as $v(x) = |\{y \mid x \circ y \text{ or } y \circ x \text{ are defined}\}|$. It can be seen that this definition is analogous to the definition of valency of a vertex in a graph.

If the law of composition is defined on any couple of elements of E, we say that the groupoid is a complete groupoid, or simply a groupoid. The groupoid is a special case of the p-groupoid.

If given x and y of E, we write $x \circ y$ without further qualification, it means that $x \circ y$ is defined.

A graph $G(V, \Gamma(V))$, where V is the set of vertices, $\Gamma(V)$ the set of edges of the graph, is called the transformation graph of a left-neutral p-groupoid $p(E, \circ)$ if:

(i) there exists a bijection $f: E \to V$,

(ii) $e \circ e'$ is defined if and only if there exists a unique oriented edge $(vv') \in \Gamma(V)$ with $f(e) = v$, $f(e') = v'$.

As an immediate consequence of this definition, we have

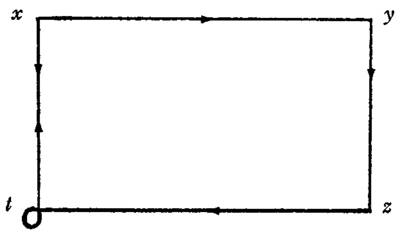
THEOREM 1. *Every graph is the transformation graph of a left-neutral p-groupoid; conversely, any left-neutral p-groupoid has a transformation graph.*

Example of left-neutral p-groupoid: The elements of E are: x, y, z, t. The rules of composition are the following:

$$x \circ y = y, \quad y \circ z = z, \quad t \circ z = z, \quad x \circ t = t, \quad t \circ x = x, \quad t \circ t = t,$$

$$z \circ y = z \circ z = z \circ t = x \circ x = y \circ y = y \circ x = z \circ x = y \circ t = t \circ y = x \circ z = \varnothing.$$

The transformation graph of $p(E, \circ)$ is the following:



A p-cycloid of length 3 is a p-groupoid such that $x_1 \circ (x_2 \circ x_3) = (x_1 \circ x_2) \circ x_3$. An element g is left-cancellable if $g \circ x = g \circ y$ implies $x = y$; a p-groupoid is left cancellable if every element is left-cancellable. A left p-semi-group is a left cancellable p-cycloid of length 3. It is immediately verified that:

PROPOSITION: Any left-neutral (complete) groupoid is a left-semi-group.

This property is interesting because the representation of the left-neutral complete groupoid is a complete graph. It agrees with the use by Hammersley of semi-groups to proceed to enumeration on complete graphs [1].

II. Factorization of Groupoids. Let us call an f-groupoid, a p-groupoid such that:

(i) $\forall x \in E,\ v(x) \geqq 1$,

(ii) $x \circ y = x' \circ y' \Rightarrow x = x',\ y = y'$,

(iii) $x \circ y \neq \emptyset \Rightarrow y \circ x = \emptyset$.

Let $p(E, \circ)$ be a p-groupoid such that $\forall x \in E,\ v(x) \geqq 1$.

$f(E_i, \circ_i)$ is a partial replica of $p(E, \circ)$ iff:

(i) $E_i \subseteq E$,

(ii) $f(E_i, \circ_i)$ is an f-groupoid,

(iii) $x \circ_i y = z \Rightarrow x \circ y = z$, but $x \circ_i y = \emptyset$ does not necessarily imply $x \circ y = \emptyset$.

Finally, we say that $p(E, \circ)$ is a simple product of p-groupoids $p(E_i, \circ_i)$ if (i) $E_i \subseteq E$ and (ii) the mappings $(x, y) \rightarrow x \circ y$ for x, y are uniquely determined by the mappings $(x, y) \rightarrow x \circ_i y$ for x, y, i. We write

$$p(E, \circ) = S \prod_i p(E_i, \circ_i).$$

Remark: We can better see the difference between the notions of direct product, sum and simple product, by means of an example.

Let $P = p(E, \circ)$ be a p-groupoid defined on $E = \{x, y, z\}$; only $x \circ y = z$, $z \circ y = y$, $y \circ z = z$ are defined. Two partial replicas are the following:

$\text{I} = p(E_1, \circ_1)$: $E_1 = \{x, y, z\}$ only $x \circ_1 y = z$ $z \circ_1 y = y$ are defined;

$\text{II} = p(E_2, \circ_2)$: $E_2 = \{y, z\}$ only $y \circ_2 z = z$ is defined.

The direct product of $p(E_1, \circ_1)$ and $p(E_2, \circ_2)$ is usually defined on the cartesian product $E_1 \times E_2$ of the underlying sets; the simple product of $p(E_1, \circ_1)$ and $p(E_2, \circ_2)$ is defined on $E = E_1 \cup E_2$.

Besides, suppose that $h: P' \rightarrow P$ is a morphism of p-groupoids. Let us choose $P' = p(E', \circ')$ where: $E' = \{y', z'\}$, only $y' \circ z' = z'$ is defined, $h(y') = y$, $h(z') = z$. Then

$$\begin{array}{ccc} P' & \rightarrow & P \\ & \searrow & \downarrow \\ & & \text{II} \end{array} \quad \text{has a sense,}$$

$$\text{while} \quad \begin{array}{ccc} & & \text{I} \\ & \nearrow & \uparrow \\ P' & \rightarrow & P \end{array} \quad \text{has no sense.}$$

The usual direct product (and sum for a similar reason) does not agree with the simple product.

The decomposition of $p(E, \circ)$ into partial replicas is not unique in general. But there is a unique maximal one obtained by choosing every $p(E_i, \circ_i)$ such that $E_i = \{x, y, z\}$, only $x \circ_i y = z$ is defined (we do not necessarily have $x \neq y \neq z$).

THEOREM 2. *If the valency of any element of a p-groupoid $p(E, \circ)$ is greater or equal to* 1, *$p(E, \circ)$ is a simple product of left-neutral p-groupoids.*

Proof. Let us first suppose that the given p-groupoid is an f-groupoid $f(E, \circ)$. We define a left-neutral p-groupoid $p(E, 1)$ by the condition: $e\ 1\ e' = e'$ iff $e \circ e'$ is defined. (This specifies the order between the factors of the product $e \circ e'$.)

Let $S(e') = \{e_i\colon\ e_i \circ e_j = e'\} \cup \{e_j\colon\ e_i \circ e_j = e'\}$. We define the left-neutral p-groupoid $p(E, 2)$ by $e'2e = e$ iff $e \in S(e')$.

$p(E, 1)$ and $p(E, 2)$ are defined in a unique way.

Conversely, $p(E, 1)$ and $p(E, 2)$ define $p(E, \circ)$ in a unique way: $e\ 1\ e' = e'$ implies that $e \circ e'$ exists, in this order. Then in $p(E, 2)$ we look for an e'' such that $e''2e' = e'$. When these conditions are satisfied they imply, by the definition of an f-groupoid, that $e'' = e \circ e'$. In that case $f(E, \circ) = S\prod p(E, 1)p(E, 2)$.

If now $p(E, \circ)$ is a general p-groupoid, it is a simple product of its partial replicas $f(E_i, \circ_i)$; hence $p(E, \circ) = S\prod_i p(E_i, 1_i)p(E_i, 2_i)$.

Example: Let us consider $P = \mathrm{I} \cdot \mathrm{II}$, where P, I, II are the p-groupoids defined in the previous paragraph.

I is the simple product of the two following left-neutral p-groupoids:

$p(E_1, 1_1)\colon E_1 = \{x, y, z\}$, only $x\ 1_1\ y = y$, $z\ 1_1\ y = y$ are defined;

$p(E_1, 2_1)\colon E_1 = \{x, y, z\}$, only $z\ 2_1\ x = x$, $z\ 2_1\ y = y$, $y\ 2_1\ z = z$, $y\ 2_1\ y = y$ are defined.

II is the simple product of the two following left-neutral p-groupoids:

$p(E_2, 1_2)\colon E_2 = \{y, z\}$, only $y\ 2_1\ z = z$ is defined;

$p(E_2, 2_2)\colon E_2 = \{y, z\}$, only $z\ 2_2\ y = y$, $z\ 2_2\ z = z$ are defined.

This paper was prepared while the author was at the University of Waterloo, Ontario, Canada, supported by the National Council of Canada.

Reference

1. J. M. Hammersley, The number of connected sets on a lattice, Theory of Graphs (International Symposium) Dunod, Paris, 1967, p. 149.